Learning equivalence relations

joint work with Ted Slaman and Tomasz Steifer

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Learning from Informant

Gold (1961) considered the following application of inductive inference:

Given a fixed sequence of languages $l_1, l_2, \ldots$ an ***informant*** secretly fixes a language l_i and at every stage s, presents the ***learner*** with a new word $w_s \in l_i$. The learner makes a guess $l(w_0, \ldots, w_s)$ to identify the presented language. They ***learn*** the language if they correctly guess in the limit, i.e., $\lim_s l(w_0, \ldots, w_s) = i$.

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Fokina-Kötzing-San Mauro (2019) looked at this in a computable structure theory setting:

InfEx learning: Given a fixed sequence of countable pairwise non-isomorphic structures $\mathcal{A}_0, \mathcal{A}_1 \dots$, the informant fixes an ***"example"*** $\mathcal{B} \cong \mathcal{A}_i$ and at stage s plays the substructure on the first s elements of $\mathcal{B}$. Again the learner makes a guess $l(\mathcal{B} \upharpoonright s)$ and learns the family $\mathcal{A}_0, \mathcal{A}_1, \dots$ if for any $\mathcal{A}_i$ and $\mathcal{B} \cong \mathcal{A}_i$, $\lim_s(\mathcal{B} \upharpoonright s) = i$.

For given vocabulary τ fix an enumeration $\varphi_i(x_0, \dots, x_i)$ of the atomic τ-formulas and let the atomic diagram of a τ-structure $\mathcal{A}$ with universe ω be

$$D(\mathcal{A})(i) = \begin{cases} 1 & \varphi_i[x_0 \dots x_i \mapsto 0 \dots i]^{\mathcal{A}} \\ 0 & \text{otherwise} \end{cases}$$

Allows us to identify structures with elements of 2^ω.

Definition
A pairwise non-isomorphic family of countable structures $\mathcal{A}_0, \dots$ is ***InfEx learnable*** if there is a function $l : 2^{<\omega} \to \omega$ such that for any i and $\mathcal{B} \cong \mathcal{A}_i$, $L(\mathcal{B}) = \lim_s l(\mathcal{B} \upharpoonright s) = i$.

An infinitary formula φ is Σ_2^{in} if it is of the form $\bigvee \exists \bar{x}_i \bigwedge \forall \bar{y}_{ij} \varphi_{ij}$ where φ_{ij} is quantifier-free.

Theorem (Bazhenov, Fokina, San Mauro 2020)
A countable sequence $\mathcal{A}_0, \dots$ is ***InfEx*** *learnable if and only if there are Σ_2^{in} formulas φ_i such that $\mathcal{A}_i \models \varphi_i$ and $\mathcal{A}_j \not\models \varphi_i$ for $i \neq j$.*

A binary relation E on a Polish space X is ***continuously (Borel) reducible*** to F on Y, $E \leq_{c(B)} F$ if there is a continuous (Borel) function $f : X \to Y$ s.t. for all $x, y \in X$, xEy iff $f(x) F f(y)$.

Theorem (Bazhenov, Cipriani, San Mauro 2023)
A countable sequence $\mathcal{A}_0, \dots$ is ***InfEx*** *learnable if and only if $\cong_{(\mathcal{A}_i)} \leq_c E_0$.*

1. The space of countable τ-structures $Mod(\tau)$ is a Polish space.
2. $Mod(\tau)/\cong$ is not countable in non-trivial cases
3. **InfEx** learnability is a property of $\cong$ on $\mathcal{A}_0, \ldots$

What about $\cong$ on the whole space or uncountable invariant subsets of $Mod(\tau)$?

Why restrict to $\cong$ and not look at other equivalence relations on Polish spaces?

The investigation of these two questions is the goal of this project.

- Consider ω with the discrete topology and 2^ω with the product topology.
- The function $L : 2^\omega \to \omega$, $L(\mathcal{A}) = \lim_s l(\mathcal{A} \upharpoonright s)$ is not continuous.
- The function $l_s : 2^\omega \to \omega$ $l_s(\mathcal{A}) = l(\mathcal{A} \upharpoonright s)$ is continuous. So, $L(\mathcal{A}) = \lim_s l_s(\mathcal{A})$.

Definition
Let E be an equivalence relation on a Polish space X and assume ω is equipped with the discrete topology. E is ***uniformly learnable***, or just ***learnable***, if there are continuous functions $l_n : X^\omega \times X \to \omega$ such that for $x \in X$ and $\vec{x} = (x_i)_{i\in\omega} \in X^\omega$, if $x \mathrel{E} x_i$ for some $i \in \omega$, then $\lim l_n(\vec{x}, x)$ exists and $x \mathrel{E} x_{L(\vec{x},x)}$ where $L(\vec{x}, x) = \lim l_n(\vec{x}, x)$.

If $x \not\mathrel{E} \vec{x}$, then L might either diverge or converge to some i. In the latter case we say that L produces a ***false positive***.

Disclaimer: To avoid dealing with effective Polish spaces and make proofs easier we will assume that we are working on 2^ω. Unless stated otherwise, all proofs work for arbitrary Polish spaces, mutatis mutandis.

For $a \in 2^\omega$ we say that a learner L is a-computable if there is an a-recursive function $f : \omega \to \omega$ such that $l_s = \Phi^a_{f(s)}$ where $(\Phi_i)_{i\in\omega}$ is a canonical enumeration of Turing operators.

Theorem (RSS)
Fix $a \in 2^\omega$. An equivalence relation E on a Polish space X is learnable by an a-computable learner if and only if it is $\Sigma^0_2(a)$.

Proof. $(\Leftarrow)$. Say E is $\Sigma^0_2(a)$, then there exists an a-recursive predicate R such that

$$x\,E\,y \iff \exists n \forall m R(x, y, n, m).$$

Define l_s by

$$l_s(\vec{x}, x) = \begin{cases} \mu i < s[(\exists n < s)(\forall m < s)R(x, x_i, m)] & \text{if such } i < s \text{ exists} \\ s & \text{otherwise} \end{cases}$$

Note that l_s is recursive in a and that $\lim_s l_s(\vec{x}, x) = \mu j[x_j\ E\ x]$, if such j exists. Otherwise $\lim_s l_s(\vec{x}, x) \uparrow$. Hence, $L = \lim_s l_s$ does not produce false positives!

$(\Rightarrow)$. Say E is learnable by an a-computable learner L and consider arbitrary $x, y \in 2^\omega$. We will extract a $\Sigma^0_2(a)$ definition using forcing. The main idea is that if we take a sufficiently mutually (a, x, y)-generic sequence $\vec{g}$, then the behaviour of the learner on $L(x, y^\frown\vec{g})$ is forced by some condition $\vec{p}$, and similarly for $L(y, x^\frown\vec{g})$. Using this $\vec{p}$ we can extract a $\Sigma^0_2(a)$ formula that is independent of x, y and defines E.

Lemma

If $g, \vec{g}$ is a sequence of sufficiently mutually generics relative to L and $L(g, \vec{g}) = k$, then there are $p, \vec{p} \prec g, \vec{g}$, such that L does not produce false positives for any $h, \vec{h} \succ p, \vec{p}$.

Proof.

Take $g, \vec{g}$ sufficiently L-generic. We have that $L(\vec{g}, g) \downarrow = k$ iff $\exists n (\forall m > n) l_m(g, \vec{g}) = k$. Let n_0 be the least such n. Then the above statement must be forced by some $p, \vec{p}$, i.e.,

$$\vec{p}, p \Vdash (\forall m > n_0) l_m(\dot{\vec{g}}, \dot{g}) \downarrow = k$$
$$\iff (\forall m > n_0)(\forall \vec{q}, q \leq \vec{p}, p)(l_m(\vec{q}, q) \downarrow \implies l_m(\vec{q}, q) = k)$$

But then in particular $L(h, h_0 \ldots h_i h h_{i+2} \ldots) = k$ where $i \neq k - 1$ and $i > |\vec{p}|$ and $h \mathrel{E} h_k$ as L cannot give false positives since $h \mathrel{E} h$. □

($\Leftarrow$). Say E is learnable by an a-computable learner L and $x \mathrel{E} y$. Take $\vec{g}$ sufficiently mutually (x, y, a)-generic and look at $L(x, y^\frown\vec{g})$, $L(y, x^\frown\vec{g})$. Say $L(x, y^\frown\vec{g}) = 0$, then by genericity x, y satisfy the following formula:

$$\exists n_0 \exists \vec{p} (\forall \vec{q} \leq \vec{p})(\forall n > n_0) \left(l_n(x, y^\frown\vec{q}) \downarrow \implies l_n(x, y^\frown\vec{q}) = 0 \right) \tag{*}$$

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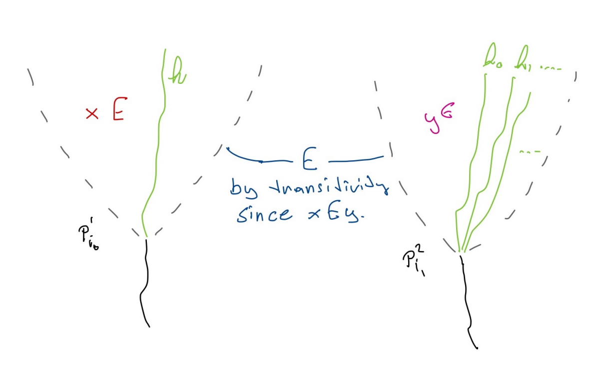
Likewise if $L(y, x^\frown \vec{g}) = 0$ then

$$\exists n_0 \exists \vec{p} (\forall \vec{q} \leq \vec{p})(\forall n > n_0) \left(l_n(y, x^\frown \vec{q}) \downarrow \implies l_n(y, x^\frown \vec{q}) = 0 \right) \qquad (**)$$

If $L(x, y^\frown \vec{g}) = i_0 \neq 0$ and $L(y, x^\frown \vec{g}) = i_1 \neq 0$, then $x \mathrel{E} g_{i_0}$, $y \mathrel{E} g_{i_1}$, and this is forced by some $\vec{p}^1$ and $\vec{p}^2$ and for any $\vec{h} \succ \vec{p}^1$, $L(x, y^\frown \vec{h}) = i_0$ (same for $\vec{h} \succ \vec{p}^2$) and $x \mathrel{E} h_{i_0}$, $y \mathrel{E} h_{i_1}$.

Look at $h \succ \vec{p}^{1}_{i_0}, \vec{h} \succ \vec{(}p^{2}_{i_1})^{\infty}$. By transitivity of E, $L(h, \vec{h}) = k$ for some k and this is again forced.



$$\begin{aligned}\exists n_0 \exists \vec{p}_0, \vec{p}_1 \exists i_0, i_1 (\forall n > n_0)(\forall \vec{q} \le \vec{p}_0)\,(L(x, y^\frown \vec{q}, n)\downarrow \implies L(x, y^\frown \vec{q}, n) = i_0) \\ \wedge(\forall \vec{q} \le \vec{p}_1)\,(L(y, x^\frown \vec{q}, n)\downarrow \implies L(y, x^\frown \vec{q}, n) = i_1) \\ \wedge \exists k (\exists r \le \vec{p}^{0}_{i_0})(\exists \vec{r} \le p^{\vec{1}}_{i_1}{}^{\infty})(\forall n > n_0)(\forall q \le r)(\forall \vec{q} \le \vec{r}) \\ (L(q, \vec{q}, n)\downarrow \implies L(q, \vec{q}, n) = k)\end{aligned} \tag{***}$$

If $x \mathrel{E} y$ then they satisfy $(*)$, $(**)$ or $(***)$. If $x \not\mathrel{E} y$, they might still satisfy $(***)$ if $L(h, \vec{h})$ gives false positives. But that cannot happen by our lemma. $(*) \vee (**) \vee (***)$ define E.

- Eventual equality on 2^ω: $x \; E_0 \; y \iff \exists m (\forall n > m) x(n) = y(n)$
- Eventual equality on $2^{\omega\omega}$: $(x_i) \; E_1 \; (y_i) \iff \exists m (\forall n > m) x_n = y_n$
- The shift action of F_2 on 2^{F_2}, $E(F_2, 2)$.

Theorem (Bazhenov, Cipriani, San Mauro)
*A countable sequence $\mathcal{A}_0, \ldots$ is **InfEx** learnable if and only if $\cong_{(\mathcal{A}_i)} \leq_c E_0$.*

- This characterization fails in our case, even for Borel reducibility, neither E_1, nor $E(F_2, 2)$ are Borel reducible to E_0.
- It follows from the Feldman-Moore theorem that for any countable Borel equivalence relation E, there is a topology such that E is $\mathbf{\Sigma^0_2}$.

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Question: Is there a universal learnable equivalence relation for continuous or Borel reducibility?

This question (with $\mathbf{\Sigma^0_2}$) seems to be open for many years.

By a result of Arnie Miller ('83), no structure can have a Σ_2^{in} Scott sentence, i.e., for no $\mathcal{A}$, $[\mathcal{A}]_{\cong}$ is $\mathbf{\Sigma}_2^0$. This implies that $\cong$ in a vocabulary τ cannot be $\mathbf{\Sigma}_2^0$.

Proposition

Let τ be a countable vocabulary. Then $Mod(\tau)$ is not learnable.

If we restrict to structures satisfying a fixed $L_{\omega_1\omega}$ sentence φ, we can find examples of learnable structures.

Example: Let φ axiomatize torsion free Abelian groups of rank 1, then $\cong_\varphi$ is learnable.

- Represent $\mathbf{\Sigma^0_2}$ relations using Borel codes. (Well-founded infinitely branching trees labeled with $\cup$, $\cap$, and codes for finite intersections of basic open sets)
- These codes can be coded by elements of 2^ω.
- How complicated is the set of codes of learnable Borel equivalence relations?

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Theorem (Louveau 1980)
If X is a recursive Polish space, $A_0, A_1 \in \Sigma_1^1$, $A_0 \cap A_1 = \emptyset$ s.t. there is $B \in \boldsymbol{\Sigma}_\alpha^0$ with $A_0 \subseteq B$ and $A_1 \cap B = \emptyset$, then B can be taken in $\Sigma_\alpha^0(HYP)$.

Lemma
If E is Δ_1^1 and learnable, then it is learnable by a hyperarithmetical learner.

Proof sketch.
By Louveau's separation theorem, if E is $\boldsymbol{\Sigma}_2^0$ and Δ_1^1, then it is $\Sigma_2^0(HYP)$. Thus by our theorem, it is learnable by a hyperarithmetical learner. □

Lemma
The set of codes of learnable Borel equivalence is Π^1_1.

Proof sketch.
T codes a learnable equivalence relation iff (1) T is well-founded, (2) its labeling is correct, and (3) there is a learner learning E_T. The first statement is Π^1_1, the second arithmetical and the third can be replaced by (3') there is a learner hyperarithmetical in T by the above Lemma. By the Spector-Gandy theorem, (3') is Π^1_1. □

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Theorem (RSS)
The set of learnable $\mathbf{\Pi^0_2}$ *equivalence relations on* 2^ω *is* $\mathbf{\Pi^1_1}$ *complete in the codes.*

Proof sketch.
Use the fact that the set of well-founded trees in $\omega^{<\omega}$ is $\mathbf{\Pi}^1_1$ complete and that $Inf = \{x \in 2^\omega : |dom(x)| = \infty\}$ is $\mathbf{\Pi}^0_2$ complete. For $x \in Inf$ let p_x be the principal function of x and define E_T as

$$xE_Ty \iff x = y \vee (p_{x_1}, p_{y_1} \in [T] \wedge x_2, y_2 \in Inf)$$

If T is well-founded, $E_T = id$. Otherwise fix $x \in [T]$, E_T can't be learnable since then

Definition

Let E be an equivalence relation on a Polish space X and assume ω is equipped with the discrete topology. E is ***uniformly Borel learnable***, or just ***Borel learnable***, if there are Borel functions $l_n : X^\omega \times X \to \omega$ such that for $x \in X$ and $\vec{x} = (x_i)_{i \in \omega} \in X^\omega$, if $x \mathrel{E} x_i$ for some $i \in \omega$, then $\lim l_n(\vec{x}, x)$ exists and $x \mathrel{E} x_{L(\vec{x},x)}$ where $L(\vec{x}, x) = \lim l_n(\vec{x}, x)$.

Borel learnability is connected to uniform learnability via the following classic fact:

Theorem

For Polish spaces $(X, \sigma), Y$ and a Borel function $f : X \to Y$, there exists a topology $\tau \supseteq \sigma$ of X such that $B((X, \tau)) = B((X, \sigma))$ and $f : X \to Y$ is τ-continuous.

Proposition

An equivalence relation E is on X is Borel learnable if and only if there exists a refinement of the topology on X such that E is uniformly learnable.

Definition
Let E be an equivalence relation on a Polish space X and assume ω is equipped with the discrete topology. We say that E is ***non-uniformly learnable***, if for every $\vec{x} = (x_i)_{i\in\omega} \in X^\omega$ there are continuous functions $l_n : X^\omega \times X \to \omega$ such that for $x \in X$, if $x \mathrel{E} x_i$ for some $i \in \omega$, then $\lim l_n(\vec{x}, x)$ exists and $x \mathrel{E} x_{L(\vec{x},x)}$ where $L(\vec{x}, x) = \lim l_n(\vec{x}, x)$.

Theorem
An equivalence relation E is non-uniformly learnable without false positives if and only if every equivalence class is $\mathbf{\Sigma^0_2}$.

Theorem
An equivalence relation E is non-uniformly learnable with false positives if and only if for every countable sequence $\vec{x}$ there is a sequence of $\mathbf{\Sigma^0_2}$ sets $(S_i)_{i\in\omega}$ such that $[x_i] \subseteq S_i$ and $[x_j] \cap S_i = \emptyset$ for $i \neq j$.