The structural complexity of models of arithmetic

joint work with Antonio Montalbán

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It is easy to answer this questions for the standard model $\mathbb{N}$: *It is structurally easy.*

But what about non-standard models?

Let us give a framework to answer this questions.

Quantifier complexity in $L_{\omega_1\omega}$

1. A formula is $\Sigma_0^{\mathrm{in}} = \Pi_0^{\mathrm{in}}$ if it is a finite quantifier free formula.
2. A formula is $\Sigma_\alpha^{\mathrm{in}}$ for $\alpha > 0$, if it is of the form $\bigvee\!\!\!\bigvee_{i\in\omega} \exists \bar{x}_i \psi_i(\bar{x}_i)$ where all $\psi_i \in \Pi_{\beta_i}^{\mathrm{in}}$ for $\beta_i < \alpha$.
3. A formula is Π_α^{in} for $\alpha > 0$, if it is of the form $\bigwedge\!\!\!\bigwedge_{i\in\omega} \forall \bar{x}_i \psi_i(\bar{x}_i)$ where all $\psi_i \in \Sigma_{\beta_i}^{\mathrm{in}}$ for $\beta_i < \alpha$.
4. $L_{\omega_1\omega} = \bigcup_{\alpha<\omega_1} \Pi_\alpha^{\mathrm{in}}$

For example, let p_n denote the (formal term) for the nth prime in PA and let $X \subseteq \omega$. Then

$$\varphi = \exists x \left(\bigwedge\!\!\!\bigwedge_{n\in X} \exists y (y \cdot p_n = x) \wedge \bigwedge\!\!\!\bigwedge_{n\notin X} \forall y (y \cdot p_n \neq x) \right)$$

is a Σ_3^{in} formula and $\mathcal{A} \models \varphi$ iff X is in the Scott set of $\mathcal{A}$.

Towards a formal framework

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For every countable structure $\mathcal{A}$ there is a sentence in the infinitary logic $L_{\omega_1\omega}$ – its Scott sentence – characterizing $\mathcal{A}$ up to isomorphism among countable structures.

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The proof heavily relies on the analysis of the α-back-and-forth relations for countable ordinals α.
The most useful definition is due to Ash and Knight:

Definition

1. $(\mathcal{A}, \bar{a}) \leq_0 (\mathcal{B}, \bar{b})$ if all atomic fromulas true of $\bar{b}$ are true of $\bar{a}$ and vice versa.
2. For non-zero $\gamma < \omega_1$, $(\mathcal{A}, \bar{a}) \leq_\gamma (\mathcal{B}, \bar{b})$ if for all $\beta < \gamma$ and $\bar{d} \in B^{<\omega}$ there is $\bar{c} \in A^{<\omega}$ such that $(\mathcal{B}, \bar{b}\bar{d}) \leq_\beta (\mathcal{A}, \bar{a}\bar{c})$.

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In an attempt to measure structural complexity, various notions of ranks have been used.

E.g. $r(\mathcal{A})$ is the least α such that for all $\bar{a}, b \in A$ if $\bar{a} \leq_\alpha \bar{b}$, then $\bar{a} \leq_\beta \bar{b}$ for all $\beta > \alpha$.

Theorem (Montalbán 2015)
The following are equivalent for countable $\mathcal{A}$ and $\alpha < \omega_1$.

1. *Every automorphism orbit of $\mathcal{A}$ is $\Sigma^{\mathrm{in}}_{\alpha}$-definable without parameters.*
2. *$\mathcal{A}$ has a $\Pi^{\mathrm{in}}_{\alpha+1}$ Scott sentence.*
3. *$\mathcal{A}$ is uniformly $\mathbf{\Delta}^0_{\alpha}$-categorical. ($\exists \Phi \exists X \forall \mathcal{B} \cong \mathcal{C} \cong \mathcal{A}(\Phi^{X \oplus (\mathcal{C} \oplus \mathcal{B})^{(\alpha)}} : \mathcal{B} \cong \mathcal{C}$)*
4. *$Iso(\mathcal{A})$ is $\mathbf{\Pi}^0_{\alpha+1}$.*
5. *No tuple in $\mathcal{A}$ is α-free.*

The least α satisfying the above is the (parameterless) Scott rank of $\mathcal{A}$.

A robust Scott rank

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Recently, an even more fine-grained notion has received some interest.

Definition
The Scott complexity of a structure $\mathcal{A}$ is the least complexity among $\Sigma^{\text{in}}_{\alpha}$, Π^{in}_{α}, and d-$\Sigma^{\text{in}}_{\alpha}$ of a Scott sentence for $\mathcal{A}$.

This notion is even more robust than the above as it corresponds to the Wadge degree of the isomorphism class of $\mathcal{A}$ (A. Miller 1983, AGH-TT).

Connection to $\leq_\alpha$

Theorem (Ash, Knight)
For two countable structures $\mathcal{A}$ the following are equivalent.

1. $(\mathcal{A}, \bar{a}) \leq_\alpha (\mathcal{B}, \bar{b})$.
2. *All $\Sigma^{\text{in}}_\alpha$ sentences true of $\bar{b}$ in $\mathcal{B}$ are true of $\bar{a}$ in $\mathcal{A}$.*
3. *All Π^{in}_α sentences true of $\bar{a}$ in $\mathcal{A}$ are true of $\bar{b}$ in $\mathcal{B}$.*

In other words, $(\mathcal{A}, \bar{a}) \leq_\alpha (\mathcal{B}, \bar{b})$ iff Π^{in}_α-$tp^{\mathcal{A}}(\bar{a}) \subseteq \Pi^{\text{in}}_\alpha$-$tp^{\mathcal{B}}(\bar{b})$.

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In other words, $(\mathcal{A}, \bar{a}) \leq_\alpha (\mathcal{B}, \bar{b})$ iff $\Pi^{\mathrm{in}}_\alpha\text{-}tp^{\mathcal{A}}(\bar{a}) \subseteq \Pi^{\mathrm{in}}_\alpha\text{-}tp^{\mathcal{B}}(\bar{b})$.

Definition
A tuple $\bar{a}$ in $\mathcal{A}$ is *α-free* if

$$\forall(\beta < \alpha)\forall\bar{b}\exists\bar{a}'\bar{b}'(\bar{a}\bar{b} \leq_\beta \bar{a}'\bar{b}' \wedge \bar{a} \not\leq_\alpha \bar{a}').$$

Definition (Makkai 1981)
The *Scott spectrum* of a theory T is the set

$$SS(T) = \{\alpha \in \omega_1 : \text{there is a countable model of } T \text{ with Scott rank } \alpha\}.$$

Here T might be a sentence in $L_{\omega_1 \omega}$.

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- Ash (1986) characterized back-and-forth relations of well-orderings. The following is a corollary: $SR(n) = 1$, $SR(\omega^\alpha) = 2\alpha$, $SR(\omega^\alpha + \omega^\alpha) = 2\alpha + 1$.
- $SS(LO) = \omega_1 - 0$
- The standard model $\mathbb{N}$ of PA has Scott rank 1: Every element is the nth successor of $\dot{0}$ for some $n \in \omega$, so the automorphism orbits are definable by $s(s(\dots(\dot{0})\dots)) = x$.
- $1 \in SS(PA)$

Formalizing back-and-forth relations

Throughout this talk $\mathcal{M}$ and $\mathcal{N}$ denote countable non-standard models of PA.

Recall that $\mathcal{M}$-finite sets can be coded by single elements, i.e., given $S \subseteq_{fin} M$ code it using $\sum_{s \in S} 2^s$. Thus finite strings $\bar{u} \in M^{<\omega}$ can be considered as the $\mathcal{M}$-finite set $\{\langle i, \bar{u}(i) \rangle : i < |\bar{u}|\}$.

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Let $Tr_{\Delta_1^0}$ be a truth predicate for bounded formulas and define the formal back-and-forth relations by induction on n:

$$\bar{u} \leq_0^a \bar{v} \Leftrightarrow \forall(x \leq a)(Tr_{\Delta_1^0}(x, \bar{u}) \rightarrow Tr_{\Delta_1^0}(x, \bar{v}))$$
$$\bar{u} \leq_{n+1}^a \bar{v} \Leftrightarrow \forall \bar{x} \exists \bar{y}\Big(|\bar{x}| \leq a \rightarrow (|\bar{y}| \leq a \wedge \bar{u}\bar{x} \leq_n^a \bar{v}\bar{y})\Big)$$

Proposition
The formal back-and-forth relations $\leq_n^x$ satisfy the following properties for all n:

1. $PA \vdash \forall \bar{u}, \bar{v}, a, b((a \leq b \wedge \bar{u} \leq_n^b \bar{v}) \rightarrow \bar{u} \leq_n^a \bar{v})$
2. $PA \vdash \forall \bar{u}, \bar{v}, a(\bar{u} \leq_{n+1}^a \bar{v} \rightarrow \bar{u} \leq_n^a \bar{v})$

Proposition
Let $\bar{a}, \bar{b} \in M$. Then $\bar{a} \leq_n \bar{b} \Leftrightarrow \forall (m \in \omega) \mathcal{M} \models \bar{a} \leq_n^{\dot{m}} \bar{b}$. Furthermore, if there is $c \in M - \mathbb{N}$ such that $\mathcal{M} \models \bar{a} \leq_n^c \bar{b}$, then $\bar{a} \leq_n \bar{b}$.

Lemma
For every $\bar{a}, \bar{b} \in M^{<\omega}$, $\bar{a} \leq_\omega \bar{b}$ *if and only if* $tp(\bar{a}) = tp(\bar{b})$.

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Recall that $\mathcal{M}$ is ***homogeneous*** if every partial elementary map $M \to M$ is extendible to an automorphism.

Lemma
If $\mathcal{M}$ is not homogeneous then $SR(\mathcal{M}) > \omega$.

Proposition
If $\mathcal{M}$ is homogeneous, then $SR(\mathcal{M}) \leq \omega + 1$.

Note that every completion T of PA has an atomic model. Take $\mathcal{M} \subseteq T$ and the subset of all Skolem terms without parameters. This is an elementary substructure and all types are isolated. By the least number principle this model is rigid and its automorphism orbits in $\mathcal{M}$ are singletons.

Theorem (Montalbán, R.)
If $\mathcal{M}$ is atomic, then $SR(\mathcal{M}) = \omega$.

Theorem (Montalbán, R.)
For any nonstandard model $\mathcal{M}$, $SR(\mathcal{M}) \geq \omega$. In particular $(1, \omega) \cap SS(PA) = \emptyset$. If $T \supseteq PA$ does not have a standard model, then $1 \notin SS(T)$.

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Definition (Harrison-Trainor, R. Miller, Montalbán 2018)
A structure $\mathcal{A} = (A, P_0^{\mathcal{A}}, \dots)$ is *infinitary interpretable* in $\mathcal{B}$ if there exists a $L_{\omega_1\omega}$ definable in $\mathcal{B}$ sequence of relations $(Dom_{\mathcal{A}}^{\mathcal{B}}, \sim, R_0, \dots)$ such that

1. $Dom_{\mathcal{A}}^{\mathcal{B}} \subseteq B^{<\omega}$,
2. $\sim$ is an equivalence relation on $Dom_{\mathcal{A}}^{\mathcal{B}}$,
3. $R_i \subseteq (B^{<\omega})^{a_{P_i}}$ is closed under $\sim$ on $Dom_{\mathcal{A}}^{\mathcal{B}}$,

and there exists a function $f_{\mathcal{B}}^{\mathcal{A}} : (Dom_{\mathcal{A}}^{\mathcal{B}}, R_0, \dots)/\sim \; \cong (A, P_0^{\mathcal{A}}, \dots)$, the *interpretation of $\mathcal{A}$ in $\mathcal{B}$*. If the formulas in the interpretation are $\Delta_\alpha^{\text{in}}$ then $\mathcal{A}$ is $\Delta_\alpha^{\text{in}}$ interpretable in $\mathcal{B}$.

Definition (Harrison-Trainor, R. Miller, Montalbán 2018)
Two structures $\mathcal{A}$ and $\mathcal{B}$ are *bi-interpretable* if there are infinitary interpretations of one in the other such that the compositions

$$f_{\mathcal{B}}^{\mathcal{A}} \circ \hat{f}_{A}^{\mathcal{B}} : Dom_{\mathcal{B}}^{Dom_{\mathcal{A}}^{\mathcal{B}}} \to \mathcal{B} \quad \text{and} \quad f_{\mathcal{A}}^{\mathcal{B}} \circ \hat{f}_{\mathcal{B}}^{\mathcal{A}} : Dom_{\mathcal{A}}^{Dom_{\mathcal{B}}^{\mathcal{A}}} \to \mathcal{A}$$

are inf. definable in $\mathcal{B}$ and $\mathcal{A}$ respectively.

Theorem (Harrison-Trainor, R. Miller, Montalbán 2018)
$\mathcal{A}$ and $\mathcal{B}$ are infinitary bi-interpretable iff their automorphism groups are Baire-measurably isomorphic.

Theorem (Harrison-Trainor, R. Miller, Montalbán 2018)
A structure $\mathcal{A}$ is Δ_α^0 interpretable in $\mathcal{B}$ iff there is a functor $F : Iso(\mathcal{B}) \to Iso(\mathcal{A})$ where the operators $\Phi : Iso(\mathcal{B}) \to Iso(\mathcal{A})$ and $\Phi_* : Hom(\mathcal{B}) \to Hom(\mathcal{A})$ are Δ_α^0.

If $\mathcal{A}$ and $\mathcal{B}$ are bi-interpretable by Δ_1^0 formulas, then $SR(\mathcal{A}) = SR(\mathcal{B})$. If that is not the case, the story is not that clear.

Theorem (Gaifman 1976)
Let T be a completion of PA and $\mathcal{L}$ a linear order. Then there is a model $\mathcal{N}_{\mathcal{L}}$ of T such that $Aut(\mathcal{N}_{\mathcal{L}}) \cong Aut(\mathcal{L})$.

- A *cut* of a model $\mathcal{M}$ is a non-empty initial segment of $\mathcal{M}$ closed under successor.
- $\mathcal{N}$ is an *end-extension* of $\mathcal{M}$ if $\mathcal{M} \preccurlyeq \mathcal{N}$ and $\mathcal{M}$ is a cut of $\mathcal{N}$.
- $\mathcal{N}$ is a minimal extension of $\mathcal{M}$ if there is no $\mathcal{K}$ with $\mathcal{M} \prec \mathcal{K} \prec \mathcal{N}$.

Theorem (Gaifman 1976)
Let $\mathcal{M}$ be any model of PA, then $\mathcal{M}$ has a minimal end extension.

$\mathcal{L}$-canonical extension

The minimal end extension is obtained by taking $\mathcal{M}(a)$, the Skolem hull of $\mathcal{M}$ with a new element a having type $p(x)$ where

- $p(x)$ is *indiscernible*: for $I \subseteq M$ with every $i \in I$ having type $p(x)$ and ordered sequences $\bar{a}, \bar{b} \in I^{<\omega}$, $tp(\bar{a}) = tp(\bar{b})$,
- $p(x)$ is *unbounded*: there is no Skolem constant c such that $x \leq c \in p(x)$.

The version of Gaifman's theorem above is obtained by taking an $\mathcal{L}$-canonical extension for given $\mathcal{L}$ over the prime model $\mathcal{N}$, i.e., take an indiscernible, unbounded type $p(x)$, and construct the model

$$\mathcal{N}_{\mathcal{L}} = \bigcup_{l_1 \leq \cdots \leq l_{|l|} \in L^{<\omega}} \mathcal{N}(l_1)(l_2)\ldots(l_{|l|})$$

This construction gives a functor $F : LO \to Mod(T)$. The functor is computable relative to T. This is equivalent to having that for any $\mathcal{L}$, $\mathcal{N}_{\mathcal{L}}$ is $\mathbf{\Delta}^0_1$ interpretable in $\mathcal{L}$.

We still need to recover $\mathcal{L}$ from $\mathcal{N}_{\mathcal{L}}$ to obtain a bi-interpretation

Mind the gap

Definition
Fix $\mathcal{M} \models PA$ and let $\mathcal{F}$ be the set of definable functions $f : M \to M$ for which $x \leq f(x) \leq f(y)$ whenever $x \leq y$. For any $a \in M$ let $gap(a)$ be the smallest set S with $a \in S$ and and if $b \in S$, $f \in \mathcal{F}$, and $b \leq x \leq f(b)$ or $x \leq b \leq f(x)$, then $x \in S$.

Define $a =_g b$ as $a =_g b \Leftrightarrow a \in gap(b)$. The gap relation partitions $\mathcal{M}$ into intervals.

Theorem (Gaifman 1976)

- *If $a \in gap(b)$ and a, b both realize the same minimal type $p(x)$, then $a = b$.*
- *$\mathcal{N}_{\mathcal{L}}/{=_g}$ is order isomorphic to $1 + \mathcal{L}$.*

So we can interpret $\mathcal{L}$ in $\mathcal{N}_{\mathcal{L}}$ using the interpretation given by

$a \in Dom^{\mathcal{L}}_{\mathcal{N}_{\mathcal{L}}} \Leftrightarrow tp(a) = p(x)$	$a \sim b \Leftrightarrow a = b$	$a \leq b \Leftrightarrow a \leq^{\mathcal{N}_{\mathcal{L}}} b$
Π^{in}_{ω}	Δ^0_1	Δ^0_1

Properties of the interpretation

- $\mathcal{N}_{\mathcal{L}}$ is Δ_1^{in} interpretable in $\mathcal{L}$
- $\mathcal{L}$ is $\Delta_{\omega+1}^{\mathrm{in}}$ interpretable in $\mathcal{N}_{\mathcal{L}}$
- $\mathcal{L}$ and $\mathcal{N}_{\mathcal{L}}$ are $\Delta_{\omega+1}^{\mathrm{in}}$ bi-interpretable
- The interpretation is ”asymmetric”.

What could be the reason for that? It turns out we can interpret a lot more than $\mathcal{N}_{\mathcal{L}}$ in $\mathcal{L}$!

Say that a type $p(\bar{x})$ consisting of formulas in some class Γ is ***sharply realized*** in $\mathfrak{K}$ if there is a structure $\mathcal{A} \in \mathfrak{K}$ and $\bar{a} \in A^{<\omega}$ such that $p(\bar{x}) = tp_{\mathcal{A}}(\bar{a}) \cap \Gamma$.

Definition

Let $\mathcal{A}$ be a structure in a vocabulary τ and let $(p_i)_{i\in\omega}$ be a listing of the sharply realized $\Pi^{\mathrm{in}}_{\alpha}$ types in $\mathcal{A}$. The ***canonical structural α-jump*** $\mathcal{A}_{(\alpha)}$ is the structure obtained by adding relation symbols $(R_i)_{i\in\omega}$ to τ such that

$$\bar{a} \in R_i^{\mathcal{A}_{(\alpha)}} \Leftrightarrow \mathcal{A} \models \bigwedge\!\!\!\bigwedge_{\varphi\in p_i} \varphi(\bar{a}).$$

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Proposition
For nonzero $\alpha, \beta < \omega_1$ then $(\mathcal{A}_{(\alpha)}, \bar{a}) \leq_{\beta} (\mathcal{A}_{(\alpha)}, \bar{b}) \Leftrightarrow (\mathcal{A}, \bar{a}) \leq_{\alpha+\beta} (\mathcal{A}, \bar{b})$. In particular $SR(\mathcal{A}) = \alpha + \beta$ iff $SR(\mathcal{A}) = \beta$.

- Recall that $\bar{a} \leq_{\alpha} \bar{b}$ iff every $\Pi^{\mathrm{in}}_{\alpha}$ formula true of $\bar{a}$ is true of $\bar{b}$.
- We showed that $\bar{a} \leq_{\omega} \bar{b}$ iff $tp(\bar{a}) = tp(\bar{b})$.

Revisiting the bi-interpretation

Theorem (Montalbán, R.)
For every completion T of PA and every linear order $\mathcal{L}$, there is $\mathcal{N}_{\mathcal{L}} \models T$ such that $\mathcal{L}$ and $(\mathcal{N}_{\mathcal{L}})_{(\alpha)}$ are Δ_1^{in} bi-interpretable.

Corollary (Montalbán, R.)
Let T be any completion of PA, then $SS(T) = \{1\} \cup \{\alpha : \omega \leq \alpha < \omega_1\}$.

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This result led us to the following result about structural jumps:

Theorem (Montalbán, R.)
The following are equivalent.

1. *$\mathcal{A}$ is Δ_1^{in} bi-interpretable with $\mathcal{B}_{(\alpha)}$.*
2. *$\mathcal{A}$ is infinitary bi-interpretable with $\mathcal{B}$ where the interpretation of $\mathcal{A}$ in $\mathcal{B}$ and $f_{\mathcal{B}}^{\mathcal{A}} \circ \tilde{f}_{\mathcal{A}}^{\mathcal{B}}$ are $\Delta_{\alpha+1}^{\mathrm{in}}$, and, the interpretation of $\mathcal{B}$ in $\mathcal{A}$ and $f_{\mathcal{A}}^{\mathcal{B}} \circ \tilde{f}_{\mathcal{B}}^{\mathcal{A}}$ are Δ_1^{in}.*

Theorem (Montalbán, R.)

1. $SS(PA) = 1 \cup \{\alpha : \omega \leq \alpha \leq \omega_1\}$
2. *If* $\mathcal{M}$ *is non-homogeneous, then* $SR(\mathcal{M}) \geq \omega + 1$.
3. *If* $\mathcal{M}$ *is non-standard atomic , then* $SR(\mathcal{M}) = \omega$.
4. *If* $\mathcal{M}$ *is non-standard homogeneous, then* $SR(\mathcal{M}) \in [\omega, \omega + 1]$.
5. *For any completion* T *of* PA*, there is a* T*-computable model* $\mathcal{M}$ *with* $SR(\mathcal{M}) = \omega_1^T + 1$.

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5. *For any completion* T *of* PA*, there is a* T*-computable model* $\mathcal{M}$ *with* $SR(\mathcal{M}) = \omega_1^T + 1$.

Thank you!