Pairs of Structures: Variations and Applications

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   joint w/ Fedor Pakhomov

Pairs of Structures

Computable structure: A structure $\mathcal{S}$ in vocabulary τ is computable if there is an algorithm that computes $R_i^{\mathcal{S}}, f_i^{\mathcal{S}}, c_i^{\mathcal{S}}$ for all $R_, f_i, c_i \in \tau$ on a computable universe S.

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$L_{\omega_1\omega}$: Extends first-order logic by allowing countable conjunctions and disjunctions.

$\Sigma_0^{\mathrm{in}} = \Pi_0^{\mathrm{in}}$...finite quantifier free formulas.

Π_α^{in} formulas: $\bigwedge_{i\in\mathbb{N}} \forall\bar{x}\theta_i(\bar{x},\bar{y}) \quad \theta_i \in \Sigma_{<\alpha}^{\mathrm{in}}$

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Scott's theorem: Every countable structure has a Scott sentence in $L_{\omega_1\omega}$ classifying it among countable structures.

Lopez Escobar theorem: The $L_{\omega_1\omega}$ definable subsets correspond with the invariant Borel subsets on $Mod(\tau)$.

Warm-up

Let $X \subseteq \omega$ be Π^0_2, i.e., $x \in X \iff \forall u \exists v R(x, u, v)$.

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Define a computable function f_R as

$$\varphi_{f_R(x)}(\langle y_1, y_2 \rangle) = \begin{cases} 1 & y_1 \leq y_2 \wedge \forall z < y_2 \exists v\, R(x, z, v) \\ 0 & y_2 < y_1 \wedge \forall z < y_2 \exists v\, R(x, z, v) \\ \uparrow & otherwise \end{cases}$$

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$$\varphi_{f_R(x)} \cong \begin{cases} \omega & x \in X \\ \{n : n \in \mathbb{N}\} & x \notin X \end{cases}$$

Let $X \subseteq \mathbb{N}$ be Σ^0_3, i.e., $x \in X \iff \exists u \forall v \exists w \, R(x, u, v, w)$.

Define a computable function g_R as

$$\varphi_{g_R(x)} = \langle 0, 0 \rangle < \langle 1, \varphi_{f_{R(0,-)}} \rangle < \langle 0, 1 \rangle < \langle 2, \varphi_{f_{R(1,-)}} \rangle < \langle 0, 2 \rangle < \dots$$

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$$\varphi_{g_R(x)} \cong \begin{cases} \{\omega \cdot n, \omega^2 : n \in \mathbb{N}\} & x \in X \\ \omega & otherwise \end{cases}$$

Warm-up

Recall $x \in X \iff \exists u \forall v \exists w R(x, u, v, w)$.

$$\varphi_{h_R(x),0} = \bullet \quad c_0 \quad \bullet \quad c_1 \quad \bullet \quad c_2 \quad \bullet \quad \cdots$$

Warm-up

Recall $x \in X \iff \exists u \forall v \exists w R(x,u,v,w)$.

$$\varphi_{h_R(x),0} = \bullet \quad c_0 \quad \bullet \quad c_1 \quad \bullet \quad c_2 \quad \bullet \quad \cdots$$

At stage s, for every $u < s$, run a strategy that activates if $|\varphi_{f_{R(x,u,-)},s}| > |\varphi_{f_{R(x,u,-)},s-1}|$:

1. If you don't have a coding location c_i, pick an unused coding location.
2. Add new element to the end of your coding location.
3. Restart all strategies for $v > u$.

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Example: $\varphi_{f_{R(x,0,-)}} \cong 5$, $\varphi_{f_{R(x,1,-)}} \cong \omega$

$$\varphi_{h_R(x)} = \bullet \blacksquare\blacksquare\blacksquare\blacksquare\blacksquare \bullet \blacksquare\blacksquare\blacksquare \bullet \blacksquare\blacksquare\blacksquare\blacksquare \bullet \blacksquare\blacksquare \bullet \blacksquare\blacksquare\blacksquare\blacksquare \bullet \blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare \cdots \bullet \blacksquare\blacksquare\blacksquare \bullet \cdots$$

$$\varphi_{h_R(x)} \cong \begin{cases} \omega \cdot 2 & x \in X \\ \omega & otherwise \end{cases}$$

We have witnessed that the isomorphism problem for $(\omega \cdot 2, \omega)$ is (Σ^0_3, Π^0_3)-hard. In fact it is complete, as "There exists a left limit point" is Σ^0_3.

1. Constructed h_R using our function f_R (for (Σ^0_2, Π^0_2)) and injury.
2. $(\omega \cdot 2, \omega)$ possessing "more structure" allows to absorb injury.
3. Π^{in}_1-th$(\omega) \subseteq \Pi^{\text{in}}_1$-th$(n)$, Π^{in}_3-th$(\omega \cdot 2) \subseteq \Pi^{\text{in}}_3$-th$(\omega)$
4. ω has a Π^{in}_3 Scott sentence, thus we can not code more in pairs involving ω.

Can we prove something general?

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Can we prove something general?

Theorem (Ash-Knight '90)
For every computable ordinal α and every $\Sigma^0_{2\alpha+1}$ $X \subseteq \omega$, there is a computable function f such that

$$\varphi_{f(x)} \cong \begin{cases} \omega^\alpha \cdot 2 & x \in X \\ \omega^\alpha & x \notin X \end{cases}.$$

Theorem (Ash-Knight '90)

For α a computable ordinal, let $\mathcal{S}_0$ and $\mathcal{S}_1$ be computable structures such that $\mathcal{S}_1 \leq_\alpha \mathcal{S}_0$ and $\{\mathcal{S}_0, \mathcal{S}_1\}$ is α-friendly. Then for any Π^0_α set X there is a computable function f such that

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$(\mathcal{S}_i)_{i \in I}$ is *α-friendly* if the structures $\mathcal{S}_i$ are uniformly computable and for $\gamma < \alpha$ $\{(i, j, \bar{a}, \bar{b}) : (\mathcal{S}_i, \bar{a}) \leq_\gamma (\mathcal{S}_j, \bar{b})\}$ is computably enumerable, uniformly in γ.

Proof and a variation

- Proved using Ash and Knight's α-system, a system for iterated priority constructions.
- Proof is prime example of a proof using systems for priority constructions, e.g., α-systems, Harrington's worker method, Montalbán's true stage machinery or game meta theorem.

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Theorem (Ash-Knight '90)
For α a computable ordinal, let $(\mathcal{S}_i)_{i\in\omega}, \mathcal{S}$ be computable structures such that $(\mathcal{S}, \mathcal{S}_i)_{i\in\omega}$ are α-friendly and for each $\beta < \alpha$ and $\bar{a} \in \mathcal{S}$, there is i and $\bar{b} \in \mathcal{S}_i$ such that Π^{in}_β-th$(\mathcal{S}, \bar{a}) \subseteq \Pi^{\mathrm{in}}_\beta$-th$(\mathcal{S}_i, \bar{b})$. Then for every Π^{in}_α set X, there exists a computable function f such that

$$\varphi_{f(x)} \cong \begin{cases} \mathcal{S} & x \in X \\ \mathcal{S}_i & \text{for some } i, \text{ if } x \notin X \end{cases}.$$

Marker '89 devised a method to extend a Σ^0_n axiomatizable theory T to a theory T' that is Σ^0_{n+1} axiomatizable, not Σ^0_n axiomatizable but preserves other model-theoretic properties.

We can do something similar using pairs of structures for $L_{\omega_1\omega}$ theories.

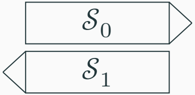
Theorem (Goncharov-Harizanov-Knight-McCoy-Miller-Solomon '05)
Let α be a computable successor ordinal and $\mathcal{S}_0, \mathcal{S}_1$ be computable α-friendly structures such that $\mathcal{S}_0 =_\beta \mathcal{S}_1$ for $\beta < \alpha$, then for any Δ^0_α set X, there is a computable function f such that

$$\varphi_{f(x)} \cong \begin{cases} \mathcal{S}_0 & x \in X \\ \mathcal{S}_1 & x \notin X \end{cases}.$$

Marker extensions using pairs of structures

Given $\mathcal{G}$ produce new structure $\mathcal{G}^{-\alpha}$ by replacing edges with copies of $\mathcal{S}_0$ and non-edges with $\mathcal{S}_1$.

$$\mathcal{G}: a \longrightarrow b \qquad \mathcal{G}^{-\alpha}: a^f$$



$$b^f$$

Formally: $\mathcal{G}^{-\alpha}$ is an $L \cup \{V/1, O/3\}$ structure where we have a bijection $f : G \to V$ and the L-reduct of $O(f(a), f(b), -)$ is isomorphic to $\mathcal{S}_0$ if aEb and $\mathcal{S}_1$ if $\neg aEb$, no L-symbol holds on elements of V and the sets V, and $O(a, b, -)$ for $a, b \in V$ are pairwise disjoint.

$\mathcal{G}^{-\alpha}$ can then be transformed to a graph using standard techniques.

If additionally every $\mathcal{S}_i$ satisfies a Π^{in}_{α} sentence not satisfied by $\mathcal{S}_{1-i}$, then we get.

1. If $\mathcal{G}$ has Scott rank β, then $\mathcal{G}^{-\alpha}$ has Scott rank $\alpha + \beta$.
2. If $\mathcal{G}$ has a copy computable in $\mathbf{d}$, then $\mathcal{G}^{-\alpha}$ has a copy computable in every $\mathbf{c}$ with $\mathbf{c}^{(\alpha)} \geq \mathbf{d}$.

Successfully used in the last two decades to lift results in computable structure theory.

Example:

Isomorphism spectrum of $\mathcal{S}$, $DgSp_{\cong}(\mathcal{S}) = \{deg(\mathcal{S}_1) : \mathcal{S}_1 \cong \mathcal{S}\}$

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Theorem

1. *(Slaman; Wehner '98) There is* $\mathcal{S}$ *with* $DgSp_{\cong}(\mathcal{S}) = \{\mathbf{d} >_T \mathbf{0}\}$.

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1. *(Slaman; Wehner '98) There is* $\mathcal{S}$ *with* $DgSp_{\cong}(\mathcal{S}) = \{\mathbf{d} >_T \mathbf{0}\}$.
2. *(GHKMMS '05) For every computable successor ordinal* α*, there is a structure with* $DgSp_{\cong}(\mathcal{S}) = non\text{-}low_{\alpha} = \{\mathbf{d} : \mathbf{d}^{(\alpha)} > \mathbf{0}^{(\alpha)}\}$.

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3. *(Greenberg-Montalbán-Slaman '13) There is $\mathcal{S}$ with $DgSp_{\cong}(\mathcal{S}) = \{\mathbf{d} : \mathbf{d} \notin HYP\}$.*

Say that distinguishing $\mathcal{S}_0$ from $\mathcal{S}_1$ is Π^0_α hard, if there is a computable operator $\Gamma : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ such that for any Π^0_α $X \subseteq 2^{\mathbb{N}}$, $\Gamma^x \cong \mathcal{S}_0$ if $x \in X$ and $\Gamma^x \cong \mathcal{S}_1$ otherwise.

Theorem (Montalbán (2nd book draft))
Let α be a computable ordinal and $\mathcal{S}_0$ and $\mathcal{S}_1$ are computable α-friendly structures with $\mathcal{S}_0 \leq_\alpha \mathcal{S}_1$, then distinguishing $\mathcal{S}_0$ from $\mathcal{S}_1$ is Π^0_α-hard.

Application: Analytic complete equivalence relations

Let E be a binary relation on a Polish space X and F be a binary relation on a Polish space Y, then E is ***Borel reducible*** to F, $E \leq_B F$ if there is a Borel $f : X \to Y$ such that for all $x_0, x_1 \in X$, $x_0 E x_1$ iff $f(x_0) F f(x_1)$.

If $E \in \Gamma$ and for every $F \in \Gamma$, $F \leq_B E$, then E is Γ-complete.

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If $E \in \Gamma$ and for every $F \in \Gamma$, $F \leq_B E$, then E is Γ-complete.

The space $Mod(\tau)$ of τ-structures with universe $\mathbb{N}$ has a natural Polish topology given by basic open sets of structures extending a finite structure.

Theorem (Louveau, Rosendal '05)
The embeddability relation on graphs is a $\boldsymbol{\Sigma}^1_1$-complete pre-order. The bi-embeddability relation on graphs is a $\boldsymbol{\Sigma}^1_1$-complete equivalence relation.

$\mathcal{S}_0$ and $\mathcal{S}_1$ are elementary bi-embeddable, $\mathcal{S}_0 \approx \mathcal{S}_1$ if and only if $\mathcal{S}_0 \preccurlyeq \mathcal{S}_1$ and $\mathcal{S}_1 \preccurlyeq \mathcal{S}_0$.
Question (SD Friedman, Moto Ros '11): Is elementary bi-embeddability on graphs $\boldsymbol{\Sigma}^1_1$ complete?

Theorem (R. '21)
Elementary embeddability on graphs is a $\mathbf{\Sigma}^1_1$-complete pre-order and elementary bi-embeddability is a $\mathbf{\Sigma}^1_1$-complete equivalence relation.

Proved by giving $F :\hookrightarrow_{Graphs} \leq_B \preccurlyeq_{Graphs}$.

1. Marker extension using pairs of structures.
2. Computably transform resulting structures into a graph using standard techniques.

Fuhrken '66 gave an example of a theory T with $2^{\aleph_0}$ non-isomorphic ***minimal models***, i.e. $\mathcal{S}_1 \preccurlyeq \mathcal{S}$ implies $\mathcal{S} = \mathcal{S}_1$.

$$T = Th((2^\omega, (x \mapsto \sigma + x)_{\sigma \in 2^{<\omega}}, (\{x : \sigma \subset x\})_{x \in 2^{<\omega}}))$$ Shelah '78

Take $\langle \bar{0} \rangle$ and $\langle \bar{1} \rangle$. They satisfy the conditions of GHKMMS theorem for $\alpha = 2$.

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Marker extension using this models lets us take F to be a computable functor with a functor $G : F(Graphs) \to Graphs$ such that F and G are pseudo-inverse ($G \circ F(\mathcal{G}) \cong \mathcal{G}$ and $F \circ G(\hat{\mathcal{G}}) \cong \hat{\mathcal{G}}$).

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Corollary (R. '21)
For every graph $\mathcal{G}$ there is a graph $\hat{G}$ such that

$$DgSp_{\approx}(\hat{\mathcal{G}}) = \{\mathbf{d} : \mathbf{d}' \in DgSp_{\sim}(\mathcal{G})\}.$$

Application: Feferman's completeness theorem

Let T be a theory given by a Σ^0_1 predicate R, i.e., $\varphi \in T \iff R(\ulcorner \varphi \urcorner)$. The ***uniform reflection principle*** for T is the theory

$$RFN(T) = T \cup \{\forall x(Prv(T, \ulcorner \varphi(\dot{x}) \urcorner) \to \varphi(x) : \varphi(x) \text{ a first-order formula}\}.$$

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Given a well-order L and a theory T we can iterate RFN along L, i.e., for $a \in L$, $RFN_{L,a}(T) = T \cup \bigcup_{b <^L a} RFN(RFN_{L,b}(T))$, and $RFN^L = \bigcup_{a \in L} RFN_{L,a}(T)$.

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Theorem (Feferman '62)
For every true arithmetical sentence φ, there is $a \in \mathcal{O}$ such that $RFN^a(\mathsf{PA}) \vdash \varphi$. Moreover, we can choose a such that $|a| < \omega^{\omega^{\omega+1}}$.

The bound in Feferman's completeness theorem is quite generous, Turing ('39) showed that for every true Π^0_1 sentence φ there is an ordinal notation a with $|a| = \omega + 1$ such that $RFN^a(\text{PA}) \vdash \varphi$ (for a broad class of reflection principles).

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Theorem (Pakhomov, R. wip)
For every true Π^0_{2n+1} sentence φ, there exists $a \in \mathcal{O}$ with $|a| = \omega^n + 1$ such that

$$RFN^a(\mathsf{PA}) \vdash \varphi.$$

Lemma

Let $n \in \omega$. Then for every Π^0_{2n+1} formula $\varphi(x)$, there is a computable function f such that ACA_0 *proves that for any i*

$$\varphi_{f(i)} \cong \begin{cases} \omega^n & \varphi(i) \\ \omega^n(1+\mathbb{Q}) & \neg\varphi(i) \end{cases}.$$

ACA_0 is the subsystem of second-order arithmetic consisting of the basic axioms, the induction scheme and the comprehension scheme for arithmetical sets.

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ACA_0 is the subsystem of second-order arithmetic consisting of the basic axioms, the induction scheme and the comprehension scheme for arithmetical sets.

Proof idea of Theorem: Let L be the linear ordering for $\varphi(0)$, then $\mathsf{ACA}_0 \vdash WO(L) \leftrightarrow \varphi$. Show that $\mathsf{ACA}_0 + WO(L)$ is conservative over $\mathsf{PA} + TI(L)$ and that $RFN(RFN_L(\mathsf{PA})) \vdash TI(L)$. It follows that $RFN(RFN_L(\mathsf{PA})) \vdash \varphi$.

Deciding whether a computable linear ordering is isomorphic to $\omega^n + 1$ is hard. The optimal Scott sentence is a computable Π^{in}_{2n+1} sentence and thus the set of indeces of isomorphic computable copies is Π^0_{2n+1} complete, as is the set of true Π^0_{2n+1} sentences.

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Theorem (Pakhomov, R. wip)
There exists a true Π^0_{2n} sentence φ such that for any computable well-ordering L of order type $\leq \omega^n$ $RFN^L(\mathsf{PA}) \nvdash \varphi$.

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