# The Borel complexity of first-order theories

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- True arithmetic is complicated: Tarski's undefinability of truth theorem, Tennenbaum's theorem, Solovay's characterization of degrees of nonstandard models, non-standard models have no finite Scott rank (Montalbán, R. 23)
- We classify the set of models of a theory using its Borel complexity.

Borel hierarchy stratifies subsets of Polish spaces. For countable $\alpha$, and $X \subseteq Mod(\tau)$

$$X \in \mathbf{\Sigma}^0_1 \iff X \text{ open} \qquad X \in \mathbf{\Pi}^0_1 \iff X \text{ closed}$$

$$X \in \mathbf{\Sigma}^0_\alpha \iff X = \bigcup_{i\in\omega}(X_i \in \mathbf{\Pi}^0_{<\alpha}) \qquad X \in \mathbf{\Pi}^0_\alpha \iff X = \bigcap_{i\in\omega}(X_i \in \mathbf{\Sigma}^0_{<\alpha})$$

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Given a countable relational vocabulary $\tau$, the set of countable $\tau$-structures with universe $\omega$ admits a canonical Polish topology.

Fix an enumeration $\varphi_i(x_0, \dots, x_i)$ of the atomic $\tau$-formulas and let the atomic diagram of a $\tau$-structure $\mathcal{A}$ with universe $\omega$ be

$$D(\mathcal{A})(i) = \begin{cases} 1 & \varphi_i[x_0 \dots x_i \mapsto 0 \dots i]^{\mathcal{A}} \\ 0 & \text{otherwise} \end{cases}$$

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$\mathcal{A} \mapsto D(\mathcal{A})$ is an homeomorphism $\mathrm{Mod}(\tau) \to 2^\omega$, giving a Polish topology on $\mathrm{Mod}(\tau)$.

$L_{\omega_1\omega}$ is similar to (finitary) first-order logic except it allows countable conjunctions and disjunctions.

For $\varphi \in L_{\omega_1\omega}$ and $\alpha$ countable

$$\varphi \in \Sigma^{\text{in}}_0 = \Pi^{\text{in}}_0 \iff \varphi \text{ finite and quantifier-free}$$
$$\varphi \in \Sigma^{\text{in}}_\alpha \iff \varphi = \bigvee\!\!\!\bigvee \exists \bar{x}_i \varphi_i \qquad , \varphi_i \in \Pi^{\text{in}}_{<\alpha}$$
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- For every $L_{\omega_1\omega}$ formula $\varphi$ there is $\alpha < \omega_1$ and $\psi \in \Sigma^{\text{in}}_\alpha$ such that $\varphi \equiv \psi$.
- (Lopez-Escobar 1969) An isomorphism invariant $X \subseteq Mod(\tau)$ is Borel iff it is $L_{\omega_1\omega}$ definable.
- (Vaught 1974) An isomophism invariant $X \subseteq Mod(\tau)$ is $\mathbf{\Pi}^0_\alpha$ iff it is $\Pi^{\text{in}}_\alpha$-definable.

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**Theorem (Keisler 1965)**
*If a finitary first-order formula $\varphi$ is equivalent to $\psi \in \Pi_n^{\mathrm{in}}$, then there is a $\forall_n$-formula $\theta$ such that $\varphi \equiv \theta$.*

Keisler proved this theorem for $L_{\infty\omega}$ using games. Harrison-Trainor and Kretschmer recently gave a new proof using forcing.

Take the formula $\varphi = \bigwedge\!\!\!\bigwedge_{\psi \in \mathrm{TA}} \psi$. Then $\mathrm{Mod}(\varphi) = Mod(\mathrm{TA}) \in \mathbf{\Pi}^0_\omega$.

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**Definition**
Let $X$ be a Polish space and $A \subseteq X$, then for any point class $\Gamma$, $A$ is *$\Gamma$-complete* if $A \in \Gamma(X)$ and for every $B \in \Gamma(Y)$ for any Polish $Y$, $B$ is *Wadge reducible* to $A$, $B \leq_W A$, i.e., there is continuous $f : Y \to X$ with $f(y) \in A$ if and only $y \in B$.

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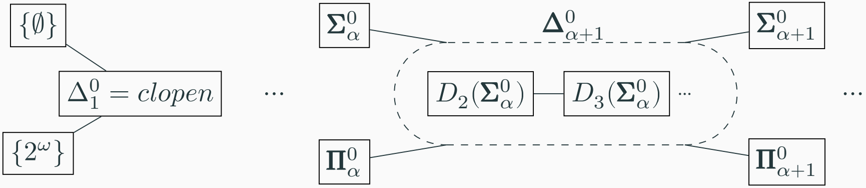

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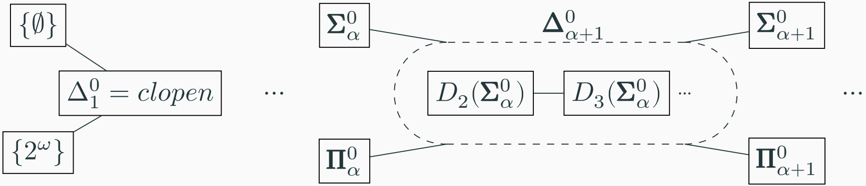


Is the complexity of a theory's set of models related to the quantifier complexity of its axiomatizations?

# First-order theories without bounded quantifier axiomatizations

**Theorem (Andrews, Gonzalez, Lempp, R., Zhu in preparation)**
*For a complete first-order theory $T$, $\mathrm{Mod}(T)$ is $\mathbf{\Pi}^0_\omega$-complete if and only if $T$ has no axiomatization by first-order formulas of bounded quantifier-complexity.*

Note that we do not need $T$ to be related to arithmetic.

This directly implies that complete theories without bounded quantifier axiomatizations can not be axiomatized by $\Pi^{\mathrm{in}}_n$ sentences for any $n$.

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**Proposition**
*There is an incomplete theory T, not axiomatizable by sentences of bounded quantifier-complexity, such that* $\mathrm{Mod}(T) \in \mathbf{\Sigma}^0_\omega$.

$(\Leftarrow)$ Say, $S$ is a set of $\exists_n$-formulas axiomatizing $\mathrm{Mod}(T)$, then $\bigwedge_{\varphi \in S} \varphi$ is $\Pi^{\mathrm{in}}_{n+1}$ and hence by Lopez-Escobar, $\mathrm{Mod}(T)$ is not $\boldsymbol{\Pi}^0_\omega$-complete.

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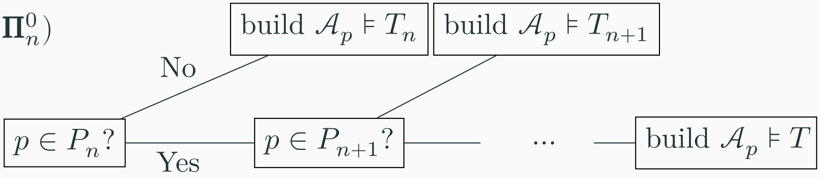
The ($\Rightarrow$) direction follows from the following lemma.

**Lemma**
*Let $T$ be a complete first-order theory for which there is a collection of theories $\{T_n\}_{n \in \omega}$ such that for all $n \in \omega$, $T \neq T_n$ but $T \cap \exists_n = T_n \cap \exists_n$. Then $\mathrm{Mod}(T)$ is $\boldsymbol{\Pi}^0_\omega$-complete. Indeed, for each $\boldsymbol{\Pi}^0_\omega$-set $P$, there is a continuous function mapping any $p \in P$ to a model of $T$, and any $p \notin P$ to a model satisfying $T_n$ for some $n \in \omega$.*

$$P = \bigcap (P_n \in \boldsymbol{\Pi}^0_n)$$



The lemma relies on a theorem due to Solovay, later generalized by Knight.

**Theorem (Solovay 1982, Knight 1999)**
*Let $T$ be a complete theory. Suppose $R \leq_T X$ is an enumeration of a Scott set $S$, with functions $t_n$ which are $\Delta^0_n(X)$ uniformly in $n$, such that for each $n$, $\lim_s t_n(s)$ is an $R$-index for $T \cap \exists_n$, and for all $s$, $t_n(s)$ is an $R$-index for a subset of $T \cap \exists_n$. Then $T$ has a model $\mathcal{B}$, representing $S$, with $\mathcal{B} \leq_T X$.*

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A ***Scott set*** $S \subseteq 2^\omega$ is a set satisfying

1. $x \leq_T y$ and $y \in S \implies x \in S$,
2. $x, y \in S \implies x \oplus y \in S$,
3. and if $x \in S$ codes an infinite binary tree $T_x$, then $S \cap [T_x] \neq \emptyset$.

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$R \in 2^\omega$ is an ***enumeration*** of a countable Scott set $S$ if $\{R^{[i]} : i \in \omega\} = S$.

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A countable model $\mathcal{M}$ ***represents*** a countable Scott set $S$ if for all complete $B_n$*-types* $\Gamma(\bar{u}, x)$ and all $\bar{c} \in M$:

$$\Gamma(\bar{c}, x) \text{ realized in } \mathcal{M} \iff \Gamma \in S \text{ and } Con(\Gamma(\bar{c}, x) \cup Diag_{el}(\mathcal{M})).$$

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- Known proofs use methods for iterated Priority constructions
- Original proof uses a Harrington style worker argument
- Version above is due to Knight (1999) and proved using version of $\alpha$-systems

Fix a theory $T$ not axiomatizable by bounded quantifier formulas and theories $T_n \neq T$ such that $T_n \cap \exists_n = T \cap \exists_n$, an enumeration $R$ of a Scott set $S$ containing $T, (T_n)$ and a Borel code $C$ for a fixed $\mathbf{\Pi}^0_\omega$ set $P = \bigcap P_n$ where $P_n$ is $\exists_n$.

In order to prove our Lemma we:

- Given $x$ produce (indices) for functions $t_n$ such that $t_n(x^{(n-1)}, s) = R(T_{n+1})$ if $x \notin P_{n,s}$ and $t_n(x^{(n-1)}, s) = R(T)$ otherwise. This can be done recursive in $x \oplus (R \oplus T \oplus \bigoplus_n T_n)' \oplus C$.
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**Corollary**
$\mathrm{Mod}(\mathrm{PA})$, *and* $\mathrm{Mod}(T)$ *for* $T$ *a completion of* $\mathrm{PA}$ *are* $\mathbf{\Pi}^0_\omega$*-complete.*

Follows from Tarski's undefinability of truth and existence of partial truth predicates. To get $T_n$ for $\mathrm{PA}$, break $\exists_n$ induction.

# Examples of theories without bounded quantifier axiomatizations

What is the role of induction in $\mathrm{PA}$ not having bounded quantifier axiomatizations? What happens with completions of $\mathrm{PA}^-$? We asked Roman Kossak who asked Ali Enayat and Albert Visser.

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**Theorem (Enayat, Visser 2023)**
*No complete sequential theory in finite vocabulary has an axiomatization by sentences of bounded quantifier complexity.*

**Definition (Pudlák 1983, Pakhomov and Visser 2022)**
A (possibly incomplete) $\tau$-theory $T$ is ***sequential*** if it admits a definitional extension to ***Adjunctive set theory*** $\mathrm{AS}(T)$, namely, in $\tau \sqcup \{\in\}$, we have the axioms

1. $\exists x\, \forall y\, (\neg y \in x)$ ("the empty set exists"), and
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In essence, sequential theories allow for coding of finite sequences as in Gödel's $\beta$-function.

*Examples of sequential theories:*
$\mathrm{PA}$, $I\Delta_0 + \exp$, $ZF$, $KP$, *even* $\mathrm{PA}^-$ (Jeřábek 2012), $\mathrm{AS} = \mathrm{AS}(\emptyset)$ (Pakhomov, Visser 2022), but *not* Robinson's $Q$ (Visser 2017).

**Theorem (Enayat, Visser 2023)**
*No complete sequential theory in finite vocabulary has an axiomatization by sentences of bounded quantifier complexity.*

The finiteness condition here is essential. Consider the Morleyization of true arithmetic (add a relation $R_\varphi$ for every formula $\varphi$). This has a compositional axiomatization in the style of Tarski's definition of satisfaction, and hence an axiomatization by $\forall_2$ formulas.

**Corollary**
*If $T$ is sequential and complete in finite vocabulary, then $\mathrm{Mod}(T)$ is $\mathbf{\Pi}^0_\omega$-complete.*

Visser will give a talk on this on *March 12 in the MOPA Seminar* (zoom)

# First-order theories with bounded axiomatizations

**Theorem (AGLRZ)**
*Let $T$ be a theory and $n \in \omega$. Then the following are equivalent.*

1. *$T$ has a $\forall_n$-axiomatization but no $\forall_{n-1}$-axiomatization.*
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- The intervals $[\mathbf{\Sigma}^0_{n-1}, \mathbf{\Pi}^0_n]$ contain infinitely many Wadge degrees as $\mathbf{\Delta}^0_n$ splits into $\aleph_1$ many degrees.
- (AGLRZ) Examples of complete $\exists_n$-axiomatizable theories of Wadge degrees $\mathbf{\Sigma}^0_n$, $D_2(\mathbf{\Sigma}^0_n)$, $\mathbf{\Pi}^0_{n+1}$ for all $n \geq 3$. We don't get $\Sigma^0_2$ and smaller degrees as:

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**Proposition (AGLRZ)**

*There is no complete consistent first-order theory $T$ such that $\mathrm{Mod}(T) \in \boldsymbol{\Sigma}^0_2$.*

The reason for this is that $\Sigma^{\mathrm{in}}_2$ sentences cannot express that structures are infinite.

**Lemma**
*Suppose* $n \in \omega$ *and* $T^+$ *and* $T^-$ *are distinct complete theories such that* $T^- \cap \exists_n \subseteq T^+ \cap \exists_n$. *Then for any* $X \in \mathbf{\Sigma}^0_n$ *there is a Wadge reduction* $f$ *such that* $f(x) \in Mod(T^+)$ *if* $x \in X$, *and* $f(x) \in \mathrm{Mod}(T^-)$ *otherwise. In particular,* $\mathrm{Mod}(T^+)$ *is* $\mathbf{\Sigma}^0_n$*-hard, and* $\mathrm{Mod}(T^-)$ *is* $\mathbf{\Pi}^0_n$*-hard.*

The proof of this lemma is similar to the proof of the core lemma for the unbounded case.

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A ***level-sentence*** set for $\mathcal{L}$ is either the set of $\exists_n$- or the set of $\forall_n$-sentences in $\mathcal{L}$ for some $n$.

**Lemma**
*Let $\Lambda$ be a level-sentence set, and let $T$ be a theory which is not $\Lambda$-axiomatizable (i.e., $\mathrm{Th}_\Lambda(T)$ does not imply $T$). Then there are complete theories $T_0, T_1$ such that $T \subseteq T_0$, $T$ is inconsistent with $T_1$, and $Th_\Lambda(T_0) \subseteq Th_\Lambda(T_1)$.*

The proof of this lemma is a compactness argument.

**Theorem (AGLRZ, cf Keisler 1965)**
*If a finitary first-order formula* $\varphi$ *is equivalent to* $\psi \in \Pi_n^{\text{in}}(L_{\infty\omega})$*, then there is a* $\forall_n$*-formula* $\theta$ *such that* $\varphi \equiv \theta$*.*

- Our proof only works for $L_{\omega_1,\omega}$, not for $L_{\infty,\omega}$.
- Keisler's proof used games
- Harrison-Trainor and Kretschmer (2023) used forcing with elementary extensions
- Our proof is much simpler, and "essentially effective"

# Effectiveness considerations

**Definition**
We say that $D$ *witnesses* the $\Gamma$-hardness of $Y \subseteq 2^\omega$ if for every Borel code $C$ for a set $X \in \Gamma$, there is a Turing operator $\Phi$ so that $\Phi^{D \oplus C \oplus p} \in Y$ if and only if $p \in X$ for every $p \in 2^\omega$. If $\Phi$ does not depend on $C$, then $D$ *uniformly witnesses* the $\Gamma$-hardness of $Y$. A Turing degree $\mathbf{d}$ *(uniformly) witnesses* the $\Gamma$-hardness of $Y$ if it contains $D$ (uniformly) witnessing the $\Gamma$-hardness of $Y$.

Our main lemmas rely on Solovay's result which was initially used to calculate the Turing degrees of models of $\mathrm{TA}$. Thus, they should be inherently effective.

**Theorem**

1. *Let $T$ be a completion of $\mathrm{PA}$. If $\mathbf{d}$ computes a non-standard model of $T$, then $\mathbf{d}$ uniformly witnesses the $\mathbf{\Pi}^0_\omega$-hardness of $\mathrm{Mod}(T)$.*
2. *Let $T$ be a completion of $\mathrm{PA}^-$. If $\mathbf{d}$ does not compute a non-standard model of $T$, then it does not witness the $\mathbf{\Sigma}^0_2$-hardness of $\mathrm{Mod}(T)$.*

If $T$ is a completion of $\mathrm{PA}$, then every Turing degree either uniformly witnesses the $\mathbf{\Pi}^0_\omega$-hardness of $\mathrm{Mod}(T)$ or fails to witness even the $\mathbf{\Sigma}^0_2$-hardness of $\mathrm{Mod}(T)$.

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Recall that $X \subseteq \omega$ is $\mathrm{PA}$ *over* $Y \subseteq \omega$ if $Y \leq_T X$ and for every $Y$-computable infinite binary tree $T$, $X$ computes a path through $T$.

**Theorem**
*Let $T$ be a theory without $\forall_n$-axiomatization and let $D$ be $\mathrm{PA}$ over $T$. Then $D$ uniformly witnesses the $\mathbf{\Sigma}^0_n$-hardness of $\mathrm{Mod}(T)$.*

Let $\exists_1^{\leq}$ be the set of bounded existential formulas in the language of $\mathrm{PA}$ and $I\exists_1^{\leq}$ the induction principle for these formulas.

**Theorem (Wilmers 1985)**
*If $\mathcal{A} \models I\exists_1^{\leq}$, then $\mathcal{A}$ is computable if and only if $\mathcal{A}$ is standard, i.e., $\mathcal{A} \cong \mathbb{N}$.*

**Theorem**
*Let $T$ be a complete consistent extension of $\mathrm{PA}^- + I\exists_1^{\leq}$. Then $\mathrm{Mod}(T)$ is $\mathbf{\Pi}^0_\omega$-complete, but $\mathbf{0}$ does not witness the $\mathbf{\Sigma}^0_2$-hardness of $\mathrm{Mod}(T)$.*

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(Shepherdson 1964) There are computable non-standard models of $\mathrm{PA}^-$.

**Question.** Is there a completion $T$ of $\mathrm{PA}^-$ such that $\mathbf{0}$ witnesses the $\mathbf{\Pi}^0_\omega$-hardness of $\mathrm{Mod}(T)$?

# Computable Structure Theory and Interactions

## Technische Universität Wien, July 15-17 2024

A workshop on computable structure theory and its interactions with other areas in logic and mathematics will take place in Vienna from July 15-17, 2024. If you are interested in attending the workshop please register here (Invited speakers do not need to register).

**Invited Speakers (preliminary, to be extended)**

- Jason Block, Brooklyn College – City University of New York
- David Gonzalez, University of California, Berkeley
- Valentina Harizanov, George Washington University
- Dariusz Kalociński, Polish Academy of Sciences
- Liling Ko, The Ohio State University
- Mateusz Łełyk, University of Warsaw
- Russell Miller, Queens College – City University of New York
- Gianluca Paolini, University of Torino
- Isabella Scott, University of Chicago
- Paul Shafer, University of Leeds
- Stefan Vatev, Sofia University
- Java Darleen Villano, University of Connecticut

**Organizers**

Vittorio Cipriani, Technische Universität Wien

Damir Dzhafarov, University of Connecticut

Ekaterina Fokina, Technische Universität Wien

Dino Rossegger, Technische Universität Wien

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Thank you!