The strong degrees of categoricity above $0''$

joint work with Barbara Csima

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A structure $\mathcal{A}$ in vocabulary τ is computable if there is an algorithm that computes $R_i^{\mathcal{A}}, f_i^{\mathcal{A}}, c_i^{\mathcal{A}}$ for all $R_, f_i, c_i \in \tau$.

Question: Given a computable structure $\mathcal{A}$, what is the least Turing degree that computes an isomorphism between all computable isomorphic copies of $\mathcal{A}$?—the ***degree of categoricity*** of $\mathcal{A}$.

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Example 1: Consider the standard model of arithmetic $\mathcal{N}$ and $\mathcal{A} \cong \mathcal{B} \cong \mathcal{N}$. We can compute an isomorphism between $\mathcal{A}$ and $\mathcal{B}$ by $\mathbf{n}^{\mathcal{A}} \mapsto \mathbf{n}^{\mathcal{B}}$ ($\mathbf{n}^{\mathcal{A}}$ is the value of the term representing n in $\mathcal{A}$.)

$\implies dgcat(\mathcal{A}) = \mathbf{0}$.

Example 2: $dgcat(\omega) = \mathbf{0}'$

(1) $\mathcal{G} = 0 \leq 1 \leq 2 \leq 3 \leq \ldots$

(2) $\mathcal{B}$ is constructed using a computable 1–1 enumeration $k_0, k_1, \ldots$ of $\emptyset'$.

$$\mathcal{B} = \quad 0 \; < \; 2 \; < \cdots < 2n < \; 2n+2 \; < \cdots$$

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(3) $Succ^{\mathcal{B}} \geq_T K$

(4) $(\forall f : \mathcal{G} \to \mathcal{B})\ f \geq_T \emptyset'$

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Observation:

- Classification up to computable isomorphism ensures equal algorithmic properties.
- Degree of categoricity $\implies$ measure of complexity. Least degree such that modulo $\mathbf{d}$, all computable copies are algorithmically equivalent.

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(3) If $dgcat(\mathcal{H}) \in \mathrm{HYP}$, then there is $f : \mathcal{G} \to \mathcal{B} \in \mathrm{HYP}$ and $f(\langle 1, 0 \rangle), f(\langle 2, 0 \rangle), \ldots$ is a HYP descending sequence in $\mathcal{B}$.

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$\implies$ $\mathcal{H}$ does not have HYP degree of categoricity.

Theorem (Csima, Franklin, Shore '13)
Every degree of categoricity is hyperarithmetic.

Thus, $\mathcal{H}$ does not have degree of categoricity.

Degrees of categoricity

- Fröhlich and Shepherdson '56 and Malt'sev '62: Computable field with non-computable transcendence basis.
- What is the least α, such that a structure $\mathbf{0}^{(\alpha)}$ computes isomorphisms between all isomorphic copies of $\mathcal{A}$?—$\mathbf{0}^{(\alpha)}$*-computable categoricity*
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Question 1: What degrees can be degrees of categoricity? Classify the degrees of categoricity.

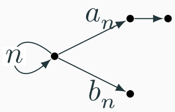
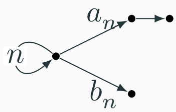
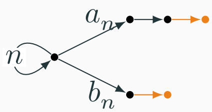
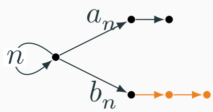
All the examples we considered had a good copy $\mathcal{G}$ and a bad copy $\mathcal{B}$ such that the isomorphisms between $\mathcal{G}$ and $\mathcal{B}$ witness the minimality of its degree of categoricity.

Definition
A degree of categoricity $\mathbf{d}$ is *strong* if there is $\mathcal{A}$ with $dgcat(\mathcal{A}) = \mathbf{d}$ and copies $\mathcal{G}$ and $\mathcal{B}$ such that for every isomorphism $f : \mathcal{G} \to \mathcal{B}$ $f \geq_T \mathbf{d}$.

Question 2: Is every degree of categoricity strong?

Fix c.e. $D \subseteq \omega$. We will construct two copies $\mathcal{G}$ and $\mathcal{B}$ of a graph.

They are the disjoint unions of the following connected components for all $n \in \omega$.

	$\mathcal{G}$	$\mathcal{B}$	
$n \notin D$			$a_n \mapsto a_n$ $b_n \mapsto b_n$
$n \searrow D$			$a_n \mapsto b_n$ $b_n \mapsto a_n$

Proposition

Every c.e. degree is a degree of categoricity.

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- Every degree d-c.e. in and above $\mathbf{0}^{(n)}$, $\mathbf{0}^{(\omega)}$ (Fokina, Kalimullin, Miller '10)

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- Every Δ_2^0 degree is a degree of categoricity. (Csima, Ng '21)

All of these examples use similar coding ideas to the one for c.e. degrees. The codings get more and more complicated and are then combined with Marker extensions (Pairs of structures).

Definition
A degree $\mathbf{d}$ is *treeable* if there exists a computable tree $T \subseteq \omega^{<\omega}$ and $f \in \mathbf{d}$ such that $f \in [T]$ and $(\forall g \in [T]) f \leq_T g$.

Theorem (Csima, R. '22)
Every strong degree of categoricity is treeable.

Proof.
Consider a pair of computable structures $\mathcal{A}_1 \cong \mathcal{A}_2$ such that $\mathbf{d}$ is the least degree computing isomorphisms between the two. Consider the tree of partial isomorphisms between $\mathcal{A}_1$ and $\mathcal{A}_2$

$$T = \{Graph_\sigma : \sigma : \mathcal{A}_1 \restriction dom(\sigma) \cong \mathcal{A}_2 \restriction rng(\sigma) \wedge dom(\sigma), rng(\sigma) \supseteq \{1 \ldots \lfloor |\sigma|/2 \rfloor\}\}$$

Then $[T] = \{f : f : \mathcal{A}_1 \cong \mathcal{A}_2\}$ and $\mathbf{d}$ is the degree of the Turing least element in $[T]$. $\square$

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Question: Is every treeable degree a degree of categoricity?

Theorem (Turetsky '20)

(1) *There is a computable structure $\mathcal{A}_1$ that has degree of categoricity $\mathbf{0}$ but high Scott rank.*

(2) *There is a computable structure $\mathcal{A}_2$ without degree of categoricity and computable dimension 2.*

A structure $\mathcal{A}$ has ***computable dimension*** $n \in (\omega \cup \{\omega\})$ if it has n computable copies up to computable isomorphism.

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$\mathcal{A}_2$ is obtained from $\mathcal{A}_1$ by a coding trick.

Idea to obtain $\mathcal{A}_1$:

(1) Given a computable tree T build a computable structure $\mathcal{A}$ such that $aut(\mathcal{A}) - id \equiv_w [T]$
(2) Force $\mathcal{A}$ to have degree of categoricity $\mathbf{0}$
(3) Take T such that $[T] \cap HYP = \emptyset$

Turetsky produces:

1. $\mathcal{A}_1$ such that $aut(\mathcal{A}_1) - id \equiv_w [Q]$ and $\{f \oplus \emptyset'' : f \in [Q]\} \equiv_w \{f \oplus \emptyset'' : f \in [T]\}$
2. $\mathcal{A}_2 \cong \mathcal{G} \cong \mathcal{B}$ such that $\{f : (f : \mathcal{G} \cong \mathcal{B})\} \equiv_w [Q]$

Getting rid of $\mathbf{0}''$

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If we were able to get rid of the $\mathbf{0}''$, then we would get that every treeable degree is a degree of categoricity using the structure $\mathcal{A}_2$.

This seems to be difficult:

- All natural structures have computable dimension 1 or ω.
- Producing something with finite computable dimension seems to rely on infinite injury.
- (Goncharov) If a structure has degree of categoricity $\leq \mathbf{0}'$, then it has computable dimension 1 or ω.

We can add the following to Turetsky's construction:

2. Force $\mathcal{A}$ to have degree of categoricity $\mathbf{0}$ (This forces $\mathcal{A}$ to code Q)

2a. For every $f \in [Q]$ $f \geq_T \emptyset''$.

Then $\{f : f : \mathcal{G} \cong \mathcal{B}\} \equiv_w [Q] \equiv_w \{f \oplus \emptyset'' : f \in [Q]\} \equiv_w \{f \oplus \emptyset'' : f \in [T]\}$.

Theorem (Csima, R.)
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Can we obtain new examples using this characterization?

$f \in \omega^\omega$ is a (Π^0_1) function singleton if there is a computable tree T with $[T] = \{f\}$.

Observation: The degree of every function singleton above $\mathbf{0}''$ is the degree of categoricity of a rigid structure of comp. dimension 2.

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Observation: The degree of every function singleton above $\mathbf{0}''$ is the degree of categoricity of a rigid structure of comp. dimension 2.

The following degrees are degrees of function singletons:

- (folklore) For all computable ordinals α, $\mathbf{0}^{(\alpha)}$ is the degree of a function singleton.
- (Jockusch, MacLaughlin '69) If $\mathbf{d}$ contains a function singleton, then so does every $\mathbf{c}$ with $\mathbf{d} \leq \mathbf{c} \leq \mathbf{d}'$.
- (Harrington '76) There is a non-arithmetical function singleton h such that $h^{(n)} \not\geq \mathbf{0}^{(\omega)}$ for all $n \in \omega$.

Corollary

(1) *For every computable* $\alpha \geq 2$, *every degree* $\mathbf{d} \in [\mathbf{0}^{(\alpha)}, \mathbf{0}^{(\alpha+1)}]$ *is a degree of categoricity.*

(2) *There is a degree* $\mathbf{d}$ *such that for every* $n \in \omega$, $\mathbf{c} \in [\mathbf{d}^{(n)}, \mathbf{d}^{(n+1)}]$ *is a non-arithmetic degree of categoricity.*

Proposition (Csima, R.)

Every degree $\mathbf{d} \in [\mathbf{0}', \mathbf{0}'']$ *is a degree of categoricity.*

Eliminating $\mathbf{0}''$ could be possible but will require new techniques:

1. (Goncharov) If a structure has degree of categoricity less than $\mathbf{0}'$, then it has computable dimension 1 or ω.
2. (Bazhenov, Yamaleev) There is a d-c.e. degree that is not the degree of categoricity of a rigid structure.

Theorem (Csima, R.)
There is a degree $\mathbf{d} \in (\mathbf{0}', \mathbf{0}'')$ that is not the degree of categoricity of a rigid structure.

We do not even know whether every function singleton is the degree of categoricity of a structure.

A degree $\mathbf{d}$ is *low for isomorphism* if whenever $\mathbf{d} \geq_T f : \mathcal{A}_1 \cong \mathcal{B}_1$ for $\mathcal{A}_1, \mathcal{B}_1$ computable, then they are computably isomorphic. (Franklin, Solomon '14)

A degree $\mathbf{d}$ is *low for paths* through Baire space if whenever $\mathbf{d} \geq_T f \in [T]$ for T in ω^ω computable, then T has a computable path.

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Theorem (Franklin, Turetsky '19)
A degree $\mathbf{d}$ is low for isomorphism if and only if it is low for paths.

All known($\star$) examples of hyperarithmetic degrees that are not degrees of categoricity can also be shown not to be treeable.

Definition
A function $f : \omega \to \omega$ has a modulus if there exists $g \gg f$ ($\forall x g(x) > f(x)$) such that if $h \gg g$, then $h \geq_T f$. It has a uniform modulus if there is a modulus $g \gg f$ and a single operator Φ such that if $h \gg g$, then $\Phi^h = g$.

Definition (Groszek-Slaman)
A modulus g for f is a (uniform) self-modulus if g is a (uniform) modulus for f and $f \geq_T g$.

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Definition (Groszek-Slaman)
A modulus g for f is a (uniform) self-modulus if g is a (uniform) modulus for f and $f \geq_T g$.

A pointclass $X \subseteq \omega^\omega$ is Π^0_n-definable if there is a Π^0_n-formula in the language of arithmetic with an additional function variable defining membership in X.

Proposition (Csima, R. 22)
Every treeable degree is a Π^0_4 singleton.

Question (Gerdes): Is there a characterization of the self-moduli in terms of arithmetic definability?

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