New Examples of Degrees of Categoricity

joint work with Barbara Csima

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Definition
A computable structure $\mathcal{A}$ is $\mathbf{d}$-*computably categorical* if for every computable $\mathcal{B} \cong \mathcal{A}$ there is a $\mathbf{d}$-computable isomorphism $f : \mathcal{A} \cong \mathcal{B}$.

If $\mathbf{d} = \mathbf{0}$, then $\mathcal{A}$ is said to be *computably categorical*.

- Investigation of computable categoricity started with work of Fröhlich and Shepherdson and Ershov in the 1960's.
- Captures the algorithmic complexity of a structure: If $\mathcal{A}$ is $\mathbf{d}$-computably categorical then mod $\mathbf{d}$, all of $\mathcal{A}$'s computable copies are computationally equivalent.

Example: Using $\mathbf{0}'$ we can compute an isomorphism between any two copies of $(\mathbb{N}, \leq)$. So, in particular given an isomorphism invariant relation R on $(\mathbb{N}, \leq)$ we can compute $R^{\mathcal{A}}$ using $\mathbf{0}'$ for any computable $\mathcal{A} \cong (\mathbb{N}, \leq)$.

Definition (Fokina, Kalimullin, R. Miller 2010)
Let $\mathcal{A}$ be a computable structure. The *categoricity spectrum of* $\mathcal{A}$ is the set

$$CatSp(\mathcal{A}) = \bigcap_{\mathcal{B}\in\mathbf{0}:\mathcal{B}\cong\mathcal{A}} \{deg(X) : \exists(f : \mathcal{A} \cong \mathcal{B}) f \leq_T X\}.$$

If $\mathbf{d}$ is the Turing least element of $CatSp(\mathcal{A})$, then $\mathbf{d}$ is the *degree of categoricity* of $\mathcal{A}$.

One of the main questions about degrees of categoricity is to characterize them. I.e.:

- Which Turing degrees are degrees of categoricity?
- Which Turing degrees can not be degrees of categoricity?

Some Examples

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- Csima, Franklin, Shore '13: Every degree of categoricity is hyperarithmetic.
- Csima, Deveau, Harrison-Trainor, Mahmoud '18: Every degree c.e. in and above $\mathbf{0}^{(\alpha)}$ for α a computable limit is a degree of categoricity.

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- Csima, Stephenson ’18: There is a degree of categoricity that is not in any interval $[\mathbf{0}^{(\alpha)}, \mathbf{0}^{(\alpha+1)}]$.
- Csima, Ng ’21: Every Δ^0_2 degree is a degree of categoricity.

What about degrees $\mathbf{d} \in [\mathbf{0}^{(\alpha)}, \mathbf{0}^{(\alpha+1)}]$ for $\alpha > 1$, in particular what if α is a limit?

Csima and Ng ’21 conjectured that all such degrees are degrees of categoricity.

Especially the limit case seems to require new techniques.

Turetsky's results

The *Scott rank* of a structure $\mathcal{S}$ is the least α such that $\mathcal{S}$ has a $\Sigma_{\alpha+1}$ Scott sentence. There are computable structures having high Scott rank, i.e., (α is not a computable ordinal).

Theorem (Turetsky '20)
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Theorem (Turetsky '20)
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Definition
A computable structure $\mathcal{S}$ has *computable dimension* $\alpha \in \{1, \ldots, \omega\}$ if it has α computable copies up to computable isomorphism.

Theorem (Turetsky '20)
There is a computable structure with computable dimension 2 that is not hyperarithmetically categorical.

The main Lemma

Looking at Turetsky's construction one can extract the following.

Lemma (cf. Turetsky '20)
Let $T \subseteq \omega^{<\omega}$ be a computable tree. Then there is a computable, computably categorical structure $\mathcal{S}_T$ such that $Aut(\mathcal{S}_T) - \{id\}$ and $[T]$ are Muchnik equivalent modulo $\mathbf{0}''$. I.e.:

$$\{\nu \oplus \emptyset'' : \nu \in (Aut(\mathcal{S}_T) - \{id\})\} \equiv_w \{f \oplus \emptyset'' : f \in [T]\}.$$

In particular, $|[T]| = 1$ if and only if $|Aut(\mathcal{S}_T) - \{id\}| = 1$.

- To obtain the first theorem take T such that $[T]$ is not Δ^1_1.
- The second theorem is obtained by modifying $\mathcal{S}_T$ slightly.

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- To obtain the first theorem take T such that $[T]$ is not Δ^1_1.
- The second theorem is obtained by modifying $\mathcal{S}_T$ slightly.
- If we can eliminate the $\mathbf{0}''$ then one could get results about degrees of categoricity. E.g.:

Dream
Let $\mathbf{d}$ be a Π^0_1 singleton. Then there is a rigid structure with computable dimension 2 and degree of categoricity $\mathbf{d}$.

The reality

- Structures of finite computable dimension other than 1 are not natural.
- In particular, if a structure is $\mathbf{0}'$ computably categorical, then it has dimension 1 or ω. (Goncharov)
- There is a d-c.e. degree that is not the degree of categoricity of any rigid structure (Bazhenov, Yamaleev ’17)

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We might still have a chance if $\mathbf{d}$ is a Π^0_1 singleton and $\mathbf{d} > \mathbf{0}'$? Nope.

Theorem (Csima, R.)
There is a degree $\mathbf{d}$, $\mathbf{0}' < \mathbf{d} < \mathbf{0}''$ that is not the degree of categoricity of a rigid structure.

Definition (Bazhenov, Kalimullin, Yamaleev '16)
The *spectral dimension* of a computable structure $\mathcal{S}$ is the least $k \leq \omega$ for which there exists a sequence of computable structures $(\mathcal{A}_i, \mathcal{B}_i)_{i\in\omega}$ such that $\mathcal{A}_i \cong \mathcal{B}_i \cong \mathcal{S}$ and

$$CatSpec(\mathcal{S}) = \bigcap_{i<k} \{deg(X) : \exists f : \mathcal{A}_i \cong \mathcal{B}_i \,\&f \leq_T X\}$$

Not a deg. of categoricity of a rigid structure

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- (BKY '16) If $\mathcal{S}$ is rigid then it has finite spectral dimension.
- Say $\mathcal{S}$ is rigid. Then there exists exactly one isomorphism between any two copies of $\mathcal{A}$.
- Bazhenov and Yamaleev used a finite injury argument to construct 2-c.e. set D such that for all $(\mathcal{A}_e, \mathcal{B}_e)$ if there exists $g : \mathcal{A}_e \cong \mathcal{B}_e$, $g \equiv_T D$, then there is $\mathcal{N}_e \cong \mathcal{A}_e : f \not\leq_T D$.
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Our result is obtained by combining true stage techniques with their argument to obtain a set $D = \emptyset' \oplus \hat{D}$ with the above properties.

The main theorem

Theorem (Csima, R.)
Let $\mathbf{d} \geq \mathbf{0}''$ *contain a* Π^0_1 *singleton, then it is the degree of categoricity of a rigid structure with comp. dimension* 2.

Corollary. Every degree $\mathbf{d} \in [\mathbf{0}^{(2+\alpha)}, \mathbf{0}^{(2+\alpha+1)}]$ for α computable is the degree of categoricity of a rigid structure with comp. dimension 2.

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Lemma (Csima, R.)
Let $T \subseteq \omega^{<\omega}$ *be a computable tree. Then there is a computable, computably categorical structure* $\mathcal{S}_T$ *such that*

$$\{\nu : \nu \in (Aut(\mathcal{S}_T) - \{id\})\} \equiv_w \{f \oplus \emptyset'' : f \in [T]\}.$$

In particular, $|[T]| = 1$ *if and only if* $|Aut(\mathcal{S}_T) - \{id\}| = 1$.

The proof is an infinite injury construction. There are three parts in building $\mathcal{S}_T$:

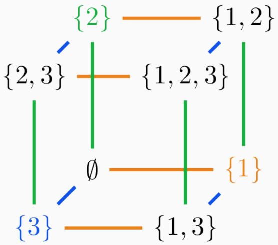
1. Coding $[T]$ into the automorphisms of $\mathcal{S}_T$
2. Making $\mathcal{S}_T$ computably categorical
3. Coding $\emptyset''$ into the automorphisms

Coding paths into automorphisms

Given a tree T we code $[T]$ into $\mathcal{S}_T$ as follows.

To every $\sigma \in \omega^{<\omega}$ we associate an infinite dimensional hypergraph with elements finite subsets of ω and an i-edge between $F, G \in [\omega]^{<\omega}$ if $F \Delta G = \{i\}$.

Formally $S_T = [\omega]^{<\omega} \times \omega^{<\omega}$, we have edge relations E_i for $i \in \omega$ and relations W_σ for $\sigma \in T$ s.t. $W_\sigma((F, \tau))$ iff $\tau = \sigma$.



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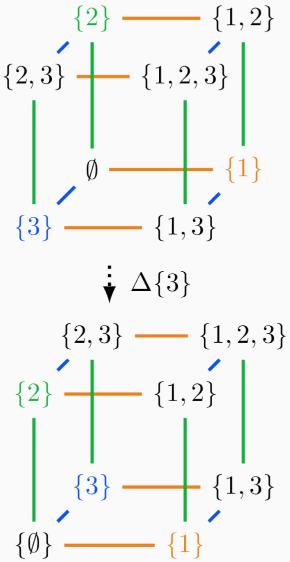
Lemma (Turetsky)
g is an automorphism of $([\omega]^{<\omega}, (E_i)_{i\in\omega})$ iff $g(F) = F \Delta H$ for fixed $H \in [\omega]^{<\omega}$.



Coding paths into automorphisms

We also have a binary predicate P such that for (F, σ), (G, τ), $P((F, \sigma), (G, \tau))$ if $\tau = \sigma^\frown i$ for some i and
$i \notin F$ and $|G|$ is even;
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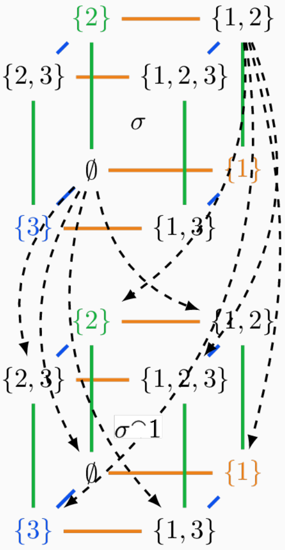
Say there is an automorphism g that acts on σ by $\Delta\{1, 2\}$. Can we deduce how g acts on $\sigma^\frown 1$? It needs to swap even with odd elements, e.g. it could act by $\Delta\{3\}$.
In particular if g acts on σ non-trivially by ΔH, then if $i \in H$ it needs to act non-trivially on $\sigma^\frown i \in T$.
Our goal is that g can act by ΔH for $H \neq \emptyset$ if and only if for every $i \in H$, $[\sigma^\frown i] \cap [T] \neq \emptyset$. To achieve this we do the following.



Coding paths into automorphisms

We add unary predicates S_n for every $n \in \omega$ and let $S_n((F,\sigma))$ for all F, n and $\sigma \in T$, and if $\sigma \in \omega^{<\omega} - T$ we let $S_n((F,\sigma))$ for all $n \in \omega$ if and only if $F \neq \emptyset$.

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- g can act by ΔH for $H \neq \emptyset$ if and only if for every $i \in H$, $[\sigma^\frown i] \cap [T] \neq \emptyset$.
  In particular, if $|[T]| = 1$, then there is exactly one non-trivial automorphism.
- Given $f \in [T]$ we can compute the automorphism that acts on $\sigma \prec f$ by $\Delta\{f(|\sigma|)\}$.
- Given automorphism g, compute a path f as follows: Given $f(k)$ for $k < n$ take

$$f(n) = \mu m[m \in \pi_1^2(g((\emptyset, \{f(x) : x < n\})))]$$

We have achieved the first goal of the construction.

However T is not computably categorical.

Getting $\mathcal{S}_T$ computably categorical

Getting $\mathcal{S}_T$ computably categorical follows the following idea:

If the i^{th} computable structure in the language of $\mathcal{S}_T$ is isomorphic but not computably isomorphic to $\mathcal{S}_T$, then we destroy the isomorphism by modifying $\mathcal{S}_T$ during the construction.

This requires an infinite injury construction that will result in our $\mathcal{S}_T$ not representing the paths of the tree T, but instead the paths of a tree Q, $\mathbf{0}''$ isomorphic to T.

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In particular, $|[T]| = 1$ if and only if $|Aut(\mathcal{S}_T) - \{id\}| = 1$.

Eliminating the double jump

We encode initial segments of the characteristic function of the index set FIN into the structure such that

$$\forall\sigma\Big(\forall n, F\,\mathcal{S}_T \vDash S_n((F,\sigma))\Big) \Rightarrow \Big(\sigma(i)\downarrow \Rightarrow (\sigma(i) \text{ is even } \Leftrightarrow i \in FIN)\Big)$$

(Recall that the tree coded in $\mathcal{S}_T$ is determined by the S_n, so this changes the tree $Q \cong T$ to a new tree $\hat{Q} \cong T$.)

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From Lemma to Theorem

To get the desired structure having as degree of categoricity the paths do the following.

1. Make a copy $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{S}_T$ with two additional elements a_{even}, a_{odd}
   $\mathcal{A}, \mathcal{B} \models P(a_{even}, (F, \emptyset))$ iff $|F|$ is even
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2. Add a constant c such that $c^{\mathcal{A}} = a_{even}, c^{\mathcal{B}} = a_{odd}$.

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- If $|[T]| = 1$ then $\mathcal{A}$ is rigid: The only automorphism of $\mathcal{S}_T$ acts by $\Delta\{i\}$ for some i on $\emptyset$, and it does not induce an automorphism of $\mathcal{A}$.

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- If $|[T]| = 1$ then $\mathcal{A}$ is rigid: The only automorphism of $\mathcal{S}_T$ acts by $\Delta\{i\}$ for some i on $\emptyset$, and it does not induce an automorphism of $\mathcal{A}$.
- $\mathcal{A}$ is still computably categorical.

Theorem (Csima, R.)
Let $\mathbf{d} > \mathbf{0}''$ contain a Π^0_1 singleton, then $\mathbf{d}$ is the degree of categoricity of a rigid structure with computable dimension 2.

Theorem (folklore; Jockusch, McLaughlin '69)
Every degree $\mathbf{d}$ *such that* $\mathbf{d} \in [\mathbf{0}^{(\alpha)}, \mathbf{0}^{(\alpha+1)}]$ *for some computable* α *contains a* Π^0_1 *singleton.*

Corollary (Csima, R.)
Every degree $\mathbf{d} \in [\mathbf{0}^{(2+\alpha)}, \mathbf{0}^{(2+\alpha+1)}]$ *for* α *a computable ordinal is the degree of categoricity of a rigid structure with computable dimension* 2*.*

Csima and Ng '21 showed that every Δ^0_2 degree is the degree of categoricity of a structure. So this only leaves open the question for the interval $[\mathbf{0}', \mathbf{0}'']$. Using Marker extensions such as in Fokina, Kalumullin, R. Miller '10 one gets.

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References

- Bazhenov, Nikolay A., and Yamaleev, Mars M. 2017. *"Degrees of Categoricity of Rigid Structures."* In Conference on Computability in Europe, 152–61. Springer.
- Turetsky, Dan 2020. *"Coding in the Automorphism Group of a Computably Categorical Structure."* Journal of Mathematical Logic 20 (03): 2050016. https://doi.org/10.1142/S0219061320500166.
- Csima, Barbara F. and Ng, Keng Meng *"Every Δ^0_2 Degree Is a Degree of Categoricity."* Preprint. degreesofcat21.pdf