Algorithmic aspects of left-orderings of solvable Baumslag-Solitar groups via dynamical realizations

Dino Rossegger (j.w. Meng-Che "Turbo" Ho and Khanh Le)
Technische Universität Wien

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Structure of the talk

Countable Borel equivalence relations and groups

Left-orderable groups and their dynamics

Borel complexity of $E_{lo}^{\mathsf{BS}(1,n)}$

Effective aspects

Countable Borel equivalence relations and groups

- Let X be a Polish space (e.g., 2^ω, 2^G, ω^ω). The set $\mathcal{B}(X)$ of ***Borel subsets*** of X is the smallest σ-algebra containing all open subsets of X.
- An equivalence relation E on a Polish space X ...
    - is ***Borel*** if it is a Borel subset of $X \times X$.
    - is ***countable (finite)*** if every E-class is countable (finite).
    - is ***hyperfinite*** if there are finite $E_i \subseteq E_{i+1}$ for $i \in \omega$ with $E = \bigcup E_i$.
- A function $f : X \to Y$ is Borel, if $f^{-1}(A)$ is Borel for every open $A \subseteq Y$.
- For two Borel equivalence relations E and F, E is ***Borel reducible*** to F, $E \leq_B F$ if there is Borel f such that $x\ E\ y$ if and only if $f(x)\ E\ f(y)$.

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Theory of countable Borel equivalence relations

A countable equivalence relation E is

- *smooth* if $E \leq_B id^{2^\omega}$
- *hyperfinite* iff $E \equiv_B E_0$: for $x, y \in 2^\omega$ $x \, E_0 \, y \iff \exists m (\forall n > m) x(n) = y(n)$
- *universal* if for every countable Borel F, $F \leq_B E$,

For example $E_s^{F_2}$ where for $x, y \in 2^{F_2}$, $x \, E_s^{F_2} \, y \iff (\exists g \in F_2) \forall h \, x(h) = y(gh)$, is universal.

Theorem (Harrington-Kechris-Louveau '90). A Borel equivalence relation E is either smooth or $E_0 \leq_B E$.

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Theorem (Slaman–Steel '88). Turing equivalence is not hyperfinite.

Question. Is Turing equivalence complete?

Theorem (Feldman–Moore '77). A countable equivalence relation E on X is Borel iff there is a countable group G such that E is the orbit equivalence relation of a Borel action of G on X ($G \curvearrowright X$).

A countable group G is ***amenable*** if there is a left-invariant, finitely additive probability measure on 2^G.

Conjecture (Weiss). If E is the orbit equivalence relation of a Borel action of a countable amenable group, then E is hyperfinite.

Verified only for a subclass. Most recently by Conley–Jackson–Marks–Seward–Tucker-Drob '23.

Left-orderable groups and their dynamics

A group G is *left-orderable* if there is a linear ordering $\leq$ on G such that for all $f, g, h \in G$ $g \leq h \implies fg \leq fh$. If in addition $g \leq h \implies gf \leq hf$, then G is bi-orderable.

$\leq$ partitions G into the *positive cone* $P = \{g \in G : g \geq id\}$, $P^{-1} = \{g^{-1} : g \in P^+\}$ and $\{id\}$.

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$LO(G) = \{P \subseteq G : P \text{ a positive cone}\}$ is a closed subspace of 2^G and thus Polish.

Let ${E_{lo}}^G$ be the orbit relation of $G \curvearrowright LO(G)$ via conjugation, i.e., $(g, x) \mapsto x^g = g^{-1}xg$.

Calderoni–Clay: Study the Borel complexity of ${E_{lo}}^G$ for countable groups.

Calderoni and Clay gave several examples of groups where E_{lo} is smooth, hyperfinite, or universal.

- (Calderoni–Clay '22) $E_{lo}^{F_2}$ is universal for $n > 2$.
- If G is torsion-free abelian, then E_{lo}^{G} is smooth.
- (Calderoni–Clay '23) $E_{lo}^{\mathrm{BS}(1,n)}$ is not smooth for $n > 1$ where $\mathrm{BS}(1,n) = \langle a, b : b^{-1}ab = a^n \rangle$

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Question (Calderoni–Clay '23) Is $E_{lo}^{\mathsf{BS}(1,n)}$ hyperfinite?

Question (Calderoni–Clay '22) Are there groups such that E_{lo}^{G} is intermediate?

Theorem (Ghys '01). Let G be a countable group. Then tfae:

(1) G is left-orderable.

(2) G acts faithfully on the real line by orientation preserving homeomorphism, i.e., there is a faithful representation $D : G \to \mathrm{Homeo}_+(\mathbb{R})$.

Idea for $(2) \implies (1)$: Fix a dense sequence (x_i) in $\mathbb{R}$ and define P_D as $g \in P_D$ if for the least i such that $D(g)(x_i) \neq x_i$, $D(g)(x_i) > x_i$.

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$(1) \implies (2)$ Fix an enumeration (g_i) of G and define a map $t : G \to \mathbb{R}$ that preserves $\leq$ by $t(g_0) = 0$ and

$$t(g_i) = \begin{cases} \max\{t(g_0), \dots t(g_{i-1})\} + 1 & \text{if } (\forall j < i) g_j \prec g_i \\ \min\{t(g_0), \dots t(g_{i-1})\} - 1 & \text{if } (\forall j < i) g_i \prec g_j \\ \frac{t(g_m)+t(g_n)}{2} & \text{if } g_i \in (g_m, g_n), m, n < i \text{ and } (\forall j < i) g_j \notin (g_m, g_n) \end{cases}$$

Let $G \curvearrowright t(G)$ via $g(t(g_i)) = t(gg_i)$. This action can be extended to obtain a faithful representation $D : G \mapsto \mathrm{Homeo}^+(\mathbb{R})$.

Note that this effectivizes:

1. There is a Turing operator Φ such that $\Phi(G, P, g) = D(g)$.
2. Similarly if we are given D, G and (x_i) we can compute a positive cone P.

Borel complexity of $E_{lo}^{\mathsf{BS}(1,n)}$

$$\mathrm{BS}(1,n) = \{a, b : b^{-1}ab = a^n\}$$

$\mathrm{BS}(1,n)$ splits over

$$1 \to \mathbb{Z}[1/n] \to \mathrm{BS}(1,n) \to \mathbb{Z} \to 1$$

i.e., it is isomorphic to $\mathbb{Z}[1/n] \rtimes \mathbb{Z}$, and thus we can represent elements as pairs (r, s), $r \in \mathbb{Z}[1/n]$, $s \in \mathbb{Z}$

Proposition. Let K and H be left-orderable groups with positive cones $P_K \subset K$ and $P_h \subset H$. Consider the short exact sequence:

$$1 \to K \to G \xrightarrow{\pi} H \to 1$$

Then $P_G = \{g \in G : \pi(g) \in P_H\} \cup P_K$ is a positive cone of G.

There are two orderings on $\mathbb{Z}[1/n]$ and $\mathbb{Z}$, the canonical one and its reverse ordering. We thus get four bi-orderings:

$$
\begin{aligned}
P_\infty^{++} &= \{(r,s) : s > 0 \vee (s = 0 \wedge r > 0)\} \\
P_\infty^{+-} &= \{(r,s) : s > 0 \vee (s = 0 \wedge r < 0)\} \\
P_\infty^{-+} &= \{(r,s) : s < 0 \vee (s = 0 \wedge r > 0)\} \\
P_\infty^{--} &= \{(r,s) : s < 0 \vee (s = 0 \wedge r < 0)\}
\end{aligned}
$$

These four are bi-orderings and conjugation invariant.

BS$(1,n)$ acts on $\mathbb{R}$ via orientation preserving affine transformations, i.e. there is $\rho :$ BS$(1,n) \to Aff^+(\mathbb{R})$ with

$$\rho(a)(x) = x + 1 \quad \text{and} \quad \rho(b)(x) = x/n.$$

I.e., $\rho(a^r b^s)(x) = n^{-s}x + r$.

For $\varepsilon \in \mathbb{R} - \mathbb{Q}$

$$P_\varepsilon^+ = \{g : \rho(g)(\varepsilon) > \varepsilon\}$$
$$P_\varepsilon^- = \{g : \rho(g)(\varepsilon) < \varepsilon\}$$

For $\varepsilon \in \mathbb{Q}$

$$Q_\varepsilon^{++} = \{g : (\rho(g)(\varepsilon) > \varepsilon) \vee (\rho(g)(\varepsilon) = \varepsilon \wedge \rho(g)(\varepsilon + 1) > \varepsilon + 1)\}$$
$$Q_\varepsilon^{+-} = \{g : (\rho(g)(\varepsilon) > \varepsilon) \vee (\rho(g)(\varepsilon) = \varepsilon \wedge \rho(g)(\varepsilon + 1) < \varepsilon + 1)\}$$
$$Q_\varepsilon^{-+} = \{g : (\rho(g)(\varepsilon) < \varepsilon) \vee (\rho(g)(\varepsilon) = \varepsilon \wedge \rho(g)(\varepsilon + 1) > \varepsilon + 1)\}$$
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Theorem (Rivas '10, Deroin–Navas–Rivas '16). The above sets are all the positive cones on BS$(1, n)$.

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One can show that for any of the above cones $T^{\circ}_{\rho(g^{-1})(\varepsilon)} = (T^{\circ}_{\varepsilon})^{g}$, so it is sufficient to show that the orbit equivalence relation of $\mathrm{BS}(1, n) \curvearrowright \mathbb{R}$ is hyperfinite.

Let $x \; E_t^n \; y \iff \exists p, q \forall k \, x(p + k) = y(q + k)$ for $x, y \in n^{\omega}$. By Dougherty–Jackson–Kechris '94 this is hyperfinite.

We define a Borel reduction to E_t^n, $f : \mathbb{R} \to n^\omega$ by $x \mapsto \{x\}$ where $\{x\}$ denotes the base n expansion of the decimal part of x. Suppose that $y = \rho((r,s))(x) = n^{-s}x + r$ and multiply the equation by a large power of n to get $n^p y = n^q x + t$. But then $\{x\}(q+k) = \{y\}(p+k)$ for all k and $f(x)\ E_t^n\ f(y)$.

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On the other hand if $f(x) = \{x\}\ E_t^n\ \{y\} = f(y)$, then $n^p y = n^q + t$ for some $p, q \in \mathbb{N}$, $t \in \mathbb{Z}$. So $y = n^{q-p}x + tn^{-p}$, so $y = \rho((tn^{-p}, q - p))(x)$.

Effective aspects

While orderable groups have seen some attention in the computability theoretic setting by Downey and Kurtz '86, Solomon '01,'02, and more recently by Harrison-Trainor '18, Darbinyan '20, and Darbinyan and Steenbock '22, the interaction with dynamics seems to have not been investigated.

Below is a short showcase on how dynamical realizations can be used in this context.

- Recall that $\varepsilon \in \mathbb{R}$ is *left c.e.* if the left cut $\{q \in \mathbb{Q}_2 : q < \varepsilon\}$ is c.e. Similarly ε has Turing degree $\mathbf{d}$ if its left cut is of that degree.
- Fix a “standard” computable presentation $\mathbb{Z}[1/n] \rtimes \mathbb{Z}$.
- Given a positive cone T°_ε of BS$(1, n)$ call ε its *base point* and T° its type.

Proposition (HLR). Left-orderings of BS$(1, n)$ are Turing equivalent to their base point, uniformly in the type.

Let G be a computable left-orderable group and P a c.e. positive cone of G. Then

$$I(G) = \{e : W_e \text{ is a positive cone}\}$$
$$I(P, G) = \{e : W_e \ E_{lo} \ P\}$$

Proposition. Let G be an infinite computable group with a computable left-ordering. Then $I(G)$ is Π^0_2-complete.

Theorem (HLR).

(1) $I(P^\circ_\infty, \mathrm{BS}(1,n))$ is Π^0_2-complete.
(2) $I(P^\circ_\varepsilon, \mathrm{BS}(1,n))$ is Σ^0_3-complete for every computable $\varepsilon \in \mathbb{R} - \mathbb{Q}$.

For the proof of the proposition and (1) simply fix an index i for a cone and at s if $W_{e,s+1} \neq W_{e,s}$ let $W_{f(e),s} = W_{i,s}$, otherwise $W_{i,s} = W_{i,s-1}$. Then f reduces Inf.

The proof for (2) is a classical movable marker construction and similar to the proof that the set of left-c.e. reals is Σ^0_3 complete.

Question. What is the complexity of $I(Q^\circ_\varepsilon, \mathrm{BS}(1,n))$ for $\varepsilon \in \mathbb{Q}$?

Question. What is the complexity of $E^{\mathrm{BS}(1,n)}_{lo}$ and E^G_{lo} for other G on indices in terms of computable reducibility?

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