Analytic complete equivalence relations and their degree spectra

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Computable structure theory studies the relationship between computational and structural properties of countable structures.

Two of my favorite topics in this area are:

1. Classification problems: *How complicated to decide whether two structure are equivalent?*
2. Degree spectra: *What are the Turing degrees of structures equivalent to a given structure?*

The main goal of this research is to explore the connections between classifications problems and degree spectra of structures.

Degree spectra and classification problems

Let $\mathcal{A}$ be a countable structure in vocabulary L and E be an equivalence relation on structures in L.

Question 1. How complicated is $M_E(\mathcal{A}) = \{\mathcal{B} : \mathcal{B}\ E\ \mathcal{A}\}$?

Question 2. How complicated is $I_E(\mathcal{A}) = \{e : \phi_e = D(\mathcal{B}) \wedge \mathcal{B}\ E\ \mathcal{A}\}$?

$D(\mathcal{B}) \in 2^\omega$ denotes the *atomic diagram* of $\mathcal{B}$ in the vocabulary $L = (R_i)_{i \in I}$,

$$D(\mathcal{B})(\ulcorner R_i(\bar{b}) \urcorner) = 1 \text{ if } R_i^{\mathcal{B}}(\bar{b}) \text{ and } D(\mathcal{B})(\ulcorner R_i(\bar{b}) \urcorner) = 0 \text{ otherwise.}$$

Question 3. How complicated is the relation E on a specific class of structures?

To answer questions like Question 1 and 3 we consider the following setting:

Let L be a relational vocabulary with symbols $(R_i/a_i)_{i\in\omega}$, then

$$Mod(L) = \prod_{i\in\omega} 2^{\omega^{a_i}}$$

is a Polish space and we can develop the Borel hierarchy $(\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Delta}^0_\alpha)$, projective hierarchy $(\mathbf{\Sigma}^1_\alpha, \mathbf{\Pi}^1_\alpha, \mathbf{\Delta}^1_\alpha)$ in the usual way.

Definition
Let E be a binary relation on a Polish space X and F be a binary relation on a Polish space Y, then E is *reducible* to F if there is a function $f : X \to Y$ such that for all $x_1, x_2 \in X$

$$x_1 \, E \, x_2 \Leftrightarrow f(x_1) \, F \, f(x_2).$$

E is *Borel reducible* to F, $E \leq_B F$, if f is Borel.

If $X = Mod(L_1)$ and $Y = Mod(L_2)$, then E is *computably reducible* to F, $E \leq_c F$, if there is a Turing operator Φ such that $\Phi^{D(\mathcal{S})} = D(f(\mathcal{S}))$ for $\mathcal{S} \in Mod(L_1)$.

Definition
E is a *Γ-complete* relation if $E \in \Gamma$ and every relation in Γ is Borel reducible to E.

$\mathcal{A}$ and $\mathcal{B}$ are ***bi-embeddable***, $\mathcal{A} \approx \mathcal{B}$, if either is isomorphic to a substructure of the other.

$\mathcal{A}$ and $\mathcal{B}$ are ***elementary bi-embeddable***, $\mathcal{A} \approxeq \mathcal{B}$, if either is isomorphic to a elementary substructure of the other.

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Bi-embeddability on graphs, $\approx_G$, is a $\mathbf{\Sigma}^1_1$ complete equivalence relation.

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Theorem (R. '21)
Elementary bi-embeddability on graphs $\approxeq_G$ is a $\mathbf{\Sigma}^1_1$ complete equivalence relation.

The isomorphism spectrum of a structure, the set of Turing degrees of its isomorphic copies, is one of the classic notions studied in computable structure theory. (Knight '86)

Fokina, Semukhin, and Turetsky; Montalbán; and Yu independently suggested to study degree spectra with respect to equivalence relations.

Definition
Given an equivalence relation E on $Mod(L)$ and $\mathcal{A} \in Mod(L)$, the *degree spectrum* of $\mathcal{A}$ w.r.t E is

$$DgSp_E(\mathcal{A}) = \{X \in 2^\omega : \exists \mathcal{B}(\mathcal{B}\ E\ \mathcal{A}\ \&\ D(\mathcal{B}) \equiv_T X)\}$$

A structure $\mathcal{A}$ is ***automorphically trivial*** if there is a finite tuple $\overline{a} \in A$ such that every permutation of A that fixes $\overline{a}$ is an automorphism.

Theorem (Knight '86; Andrews, Miller '15; Fokina, R., San Mauro '19; R. '18)
If $\mathcal{A}$ is not automorphically trivial, then $DgSp_E(\mathcal{A})$ is closed upwards, otherwise it is a single Turing degree for all $E \in \{\cong$ (Knight), $\approx$ (FRS), $\approxeq$ (R.), $\equiv$ (AM)$\}$.

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	$\cong$	$\approx$	$\approxeq$	$\equiv$
$\{X : X \geq_T S\}$ for all $S \in 2^\omega$	✓(Richter '81)	✓	✓	✓
$\{X : X >_T \emptyset\}$	✓(Slaman; Wehner '98)	✓	✓	✓
$\{X : X^{(n)} >_T \emptyset^{(n)}\}$ for all $n \in \omega$	✓(GHKMMS '05)	✓	✓	✓
$\{X : X^{(\alpha)} >_T \emptyset^{(\alpha)}\}$ for all $\alpha \in \mathcal{O}, \alpha \geq \omega$	✓(GHKMMS '05)	✓	✓	✗
$\{X : X \notin \Delta^1_1\}$	✓(GMS '13)	✓	✓	✗
$\{X \geq_T S_1\} \cup \{X \geq_T S_2\}$ for $S_1 \mid_T S_2$	✗(Knight et al.)	(✓)	✗	✓
⋮	⋮	⋮	⋮	⋮

Observation: The complexity of the equivalence relation restricts the complexity of its degree spectra.

Proposition (folklore)

If E is $\mathbf{\Pi}^0_\alpha$, then for every $\mathcal{A} \in Mod(L)$, $DgSp_E(\mathcal{A})$ is $\mathbf{\Sigma}^0_{\alpha+1}$.

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Another example arises from Scott's isomorphism theorem:

Proposition (folklore)

Every isomorphism spectrum is Borel.

	$\cong$	$\approx$	$\approxeq$	$\equiv$
$\{X \geq_T S_1\} \cup \{X \geq_T S_2\}$ for $S_1 \mid_T S_2$	✗(Knight et al.)	(✓)	✗	(✓)

Theorem (Harrison-Trainor ’22 (wip))
There are sets $S_1 \mid_T S_2$ such that $\{X \geq_T S_1\} \cup \{X \geq_T S_2\}$ is the bi-embeddability spectrum of a structure.

Theorem (Melnikov, Montalbán ’18)
Let (X, G, a) be an effective transformation group and E_G the orbit equivalence relation. Then for every $x \in X$, $DgSp_{E_G}(x) \neq \{X \geq_T S_1\} \cup \{X \geq_T S_2\}$ for any $S_1 \mid_T S_2$.

Two cones

	$\cong$	$\approx$	$\approxeq$	$\equiv$
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- Elementary bi-embeddability allows coding: If $\mathcal{A} \preccurlyeq \mathcal{B}$, then for all $\bar{a} \in A^{<\omega}$ $\exists - tp_{\mathcal{A}}(\bar{a}) = \exists - tp_{\mathcal{B}}(\bar{a})$.
- Most examples of isomorphism spectra carry over.

Reducing bi-embeddability to elementary bi-embeddability

Theorem (R.)
The elementary bi-embeddability relation on graphs is $\mathbf{\Sigma}^1_1$-complete.

We prove this theorem by giving a reduction from $\hookrightarrow_G$ to $\preccurlyeq_G$. It then follows from the completeness of $\hookrightarrow_G$ (Louveau, Rosendal) that $\preccurlyeq_G$ is $\mathbf{\Sigma}^1_1$ complete. That $\approxeq_G$ is $\mathbf{\Sigma}^1_1$ complete is an immediate corollary.

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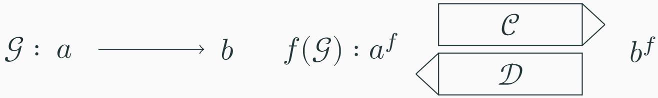
We do a Marker extension (pairs of structures technique) using structures with a special model theoretic property to obtain a result about degree spectra.

Theorem (R.)
Let $\mathcal{G}$ be a graph, then there exists a graph $\hat{\mathcal{G}}$ such that

$$DgSp_{\approxeq}(\hat{\mathcal{G}}) = \{X : X' \in DgSp_{\approx}(\mathcal{G})\}.$$

Given $\mathcal{G}$ we first produce a structure $f(\mathcal{G})$ by replacing edges with copies of a L—structure $\mathcal{C}$ and non-edges with copies of $\mathcal{D}$.



Formally: $f(\mathcal{G})$ is an $L \cup \{V/1, O/3\}$ structure where we have a bijection $f : G \to V$ and the L-reduct of $O(f(a), f(b), -)$ is isomorphic to $\mathcal{C}$ if aEb and $\mathcal{D}$ if $\neg aEb$, no L-symbol holds on elements of V and the sets V, and $O(a, b, -)$ for $a, b \in V$ are pairwise disjoint.

Proof sketch

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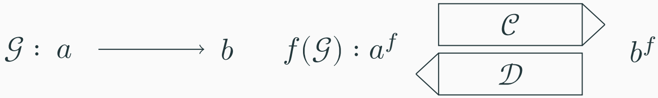

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If $h : \mathcal{G}_1 \hookrightarrow \mathcal{G}_2$, then there is an induced embedding $f(h) : f(\mathcal{G}_1) \hookrightarrow f(\mathcal{G}_2)$. To show that $f(h)$ is elementary we show that player II has a winning strategy in the Ehrenfeucht-Fraïssé games $G_n((f(\mathcal{G}_1), \bar{a}), (f(\mathcal{G}_2), f(h)(\bar{a}))$ for all n, and $\bar{a} \in f(G_1)^{<\omega}$.

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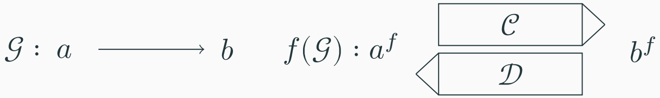

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That $\mathcal{G}_1 \hookrightarrow \mathcal{G}_2$ iff $f(\mathcal{G}_1) \preccurlyeq f(\mathcal{G}_2)$ it is sufficient that $\mathcal{C} \not\preccurlyeq \mathcal{D}$, $\mathcal{D} \not\preccurlyeq \mathcal{C}$ and $\mathcal{C} \not\equiv \mathcal{D}$.
We can transform the structures $f(\mathcal{G})$ into a graph using standard codings.

For $DgSp_{\approxeq}(f(\mathcal{G})) = \{X : X' \in DgSp_{\approx}(\mathcal{G})\}$ it is sufficient that

1. for all $\mathcal{A} \approx \mathcal{G}$ $\mathcal{A} \geq_T f(\mathcal{A})$,
2. for all $\mathcal{B} \approxeq f(\mathcal{G})$ there is $\mathcal{A}$
   2.1 with $f(\mathcal{A}) \cong \mathcal{B}$,
   2.2 and $\mathcal{B}' \geq_T \hat{\mathcal{A}} \cong \mathcal{A}$.

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2.1 is essential and non-trivial, e.g. take $\mathcal{C} = (\omega, \omega + \zeta)$, $\mathcal{D} = (\omega + \zeta, \omega)$. Then we would get that $f(\mathcal{G})' \geq_T \hat{\mathcal{G}} \cong \mathcal{G}$ but the structure obtained if we use $\mathcal{C} = (\omega, \omega) = \mathcal{D}$ would elementary embed into $f(\mathcal{G})$.

For $DgSp_{\cong}(f(\mathcal{G})) = \{X : X' \in DgSp_{\approx}(\mathcal{G})\}$ it is sufficient that

1. for all $\mathcal{A} \approx \mathcal{G}$ $\mathcal{A} \geq_T f(\mathcal{A})$,
2. for all $\mathcal{B} \cong f(\mathcal{G})$ there is $\mathcal{A}$
    2.1 with $f(\mathcal{A}) \cong \mathcal{B}$,
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1. $\mathcal{C} \equiv \mathcal{D}$,

2'. for every $\mathcal{A} \ncong \mathcal{C}$, $\mathcal{A} \not\preccurlyeq \mathcal{C}$,

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Definition

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Question (Vaught): Does every countable complete theory with a minimal model have a prime model?

Theorem (Fuhrken '66)
There is a countable complete theory with $2^{\aleph_0}$ minimal models.

Theorem (Shelah '78)
For every $\kappa \leq \aleph_0$, there is a countable complete theory with κ minimal models.

Theorem (Hjorth '96)
In L there is a countable complete theory with $\aleph_1$ many minimal models but no perfect set of minimal models.

For $\nu \in 2^{<\omega}$ define $F_\nu : 2^\omega \to 2^\omega$, $\sigma \mapsto \nu +_2 \sigma$ (where ν is interpreted as $\nu^\frown\bar{0}$ and $+_2$ is base 2 addition).

Let $R_\nu = \{\sigma \in 2^\omega : \nu \preceq \sigma\}$ and consider the theory T of

$$\mathcal{A} = (2^\omega, \langle F_\nu \rangle_{\nu \in 2^{<\omega}}, \langle R_\nu \rangle_{\nu \in 2^{<\omega}}).$$

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Shelah used T and variations of T to prove his theorem. It is easy to see that

1. T has quantifier elimination,
2. the substructure $\langle \sigma \rangle$ generated by $\sigma \in 2^\omega$ is an elementary substructure of $\mathcal{A}$,
3. $\langle \sigma \rangle$ is minimal,
4. if $\exists^\infty i\ \sigma(i) \neq \tau(i)$, then there is a Σ_2^c sentence distuingishing $\langle \sigma \rangle$ and $\langle \tau \rangle$.

$$\exists x \bigwedge_{\nu \preceq \sigma} R_\sigma(x)$$

Lemma
Let X be $\Delta^0_2(Y)$ for a set Y, then there exists a sequence of structures $(\mathcal{C}_i)_{i\in\omega}$, uniformly computable in Y, such that

$$\mathcal{C}_i \cong \begin{cases} \langle\bar{0}\rangle & \text{if } i \in X, \\ \langle\bar{1}\rangle & \text{if } i \notin X. \end{cases}$$

We do a Marker extension with $\langle\bar{0}\rangle$ and $\langle\bar{1}\rangle$ to obtain that for every graph $\mathcal{G}$, there is a graph $\hat{\mathcal{G}}$ such that

$$DgSp_{\cong}(\hat{\mathcal{G}}) = \{X : X' \in DgSp_{\approx}(\mathcal{G})\}.$$

(Harrison-Trainor '22) There are sets $S_1 \mid_T S_2$ such that $\{X \geq_T S_1\} \cup \{X \geq_T S_2\}$ is the bi-embeddability spectrum of a structure.

$\implies \exists \mathcal{G}$ with $DgSp_{\approxeq}(\mathcal{G}) = \{Y : Y' \in \{X \geq_T S_1\} \cup \{X \geq_T S_2\}\}$

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(R. '18) No elementary bi-embeddability spectrum can be the union of cones above incomparable degrees.

But $DgSp_{\approxeq}(\hat{\mathcal{G}})' = \{X \geq_T S_1\} \cup \{X \geq_T S_2\}$.

Corollary
There is an elementary bi-embeddability spectrum $\mathcal{X}$ such that $\mathcal{X}' = \{X' : X \in \mathcal{X}\}$ is not the elementary bi-embeddability spectrum of a structure.

Isomorphism spectra do jump and its an open question whether $\approx$ or $\equiv$ spectra jump.

The reduction $\approx_G \to \cong_G$ is functorial and has a pseudo-inverse:

There is a computable functor $F : (G, \hookrightarrow) \to (G, \preccurlyeq)$ and a functor $H : (F(G), \preccurlyeq) \to (G, \hookrightarrow)$ such that the $F \circ G$ and $G \circ F$ are naturally isomorphic to the identity functors.
H is not computable: Given $F(G)$ it takes one jump to decide whether a structure coding the edge relation between a_f and b_f is isomorphic to $\langle 0 \rangle$ or $\langle 1 \rangle$.

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Write $\mathcal{A} \preccurlyeq_n \mathcal{B}$ if there is an embedding $f : \mathcal{A} \to \mathcal{B}$ such that $n\text{-}tp^{\mathcal{A}}(\bar{a}) = n\text{-}tp^{f(\mathcal{A})}(f(\bar{a}))$ for all $\bar{a} \in A^{\omega}$ and $\cong_n$ for the induced equivalence relation.

Corollary
The $\cong_n$ relation on graphs is a $\mathbf{\Sigma}^1_1$-complete equivalence relation for all n.

Corollary
There is no computable functor $F : (G, \hookrightarrow) \to (G, \preccurlyeq)$ with computable (continuous) pseudo-inverse.

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Thank you!