The structural complexity of models of arithmetic

joint work with Antonio Montalbán

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Goal: We would like to measure how complicated countable mathematical structures are structurally.

Examples of structures: Groups, Rings, Linear orderings, Vector spaces,...

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Structural complexity of countable models

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In this project we are interested in the structural complexity of models of arithmetic.

Peano arithmetic PA are first-order axioms for arithmetic in the vocabulary $(0, 1, +, =, \cdot)$.

Discrete Semiring axioms:

$$\forall x(0 \neq 0 + 1) \qquad \forall x, y(x + 1 = x + 1 \implies x = y)$$

$$\forall x(x + 0 = x) \qquad \forall x, y(x + (y + 1) = (x + y) + 1)$$

$$\forall x(x \cdot 0 = 0) \qquad \forall x, y(x \cdot S(y) = x \cdot y + x)$$

Induction schema: For all first order formulas in the language of arithmetic

$$\forall \bar{y}\,((\varphi(0, \bar{y}) \wedge \forall x\,(\varphi(x, \bar{y}) \implies \varphi(x + 1, \bar{y}))) \implies \forall x(\varphi(x, \bar{y})))$$

The *standard model* of arithmetic is the structure $\mathcal{N} = (\mathbb{N}, 0, 1, +, \cdot)$.

Peano arithmetic is a first-order theory (quantification over elements of the universe) and thus suffers from the usual first-order theoretic limitations.

Theorem (Gödel '31, First incompleteness theorem)

Every consistent recursive first-order theory T in which we can formalize PA is incomplete, i.e., there are sentences φ such that neither φ nor $\neg\varphi$ are provable from T.

(We assume PA is consistent)

1. There are first-order sentences φ true about $\mathcal{N}$ that are not provable in PA.
2. True arithmetic, $Th(\mathcal{N})$, is only one possible completion. There are $\mathcal{S} \models \mathrm{PA}$ such that $\mathcal{S} \models \varphi$ while $\mathcal{N} \models \neg\varphi$.

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Theorem (Gödel '30, Compactness theorem)

Let X be a countable set of first-order sentences. If every finite subset of X has a model, then X has a model.

Add a new constant symbol c to the vocabulary of PA and take the set $X = \{c \geq \mathbf{n} : n \in \mathbb{N}\}$. Take $Y \subseteq_{finite} Th(\mathcal{N}) \cup X$. Then $(\mathcal{N}, c) \models Y$ with c larger then the largest constant in Y. So, by compactness, $Th(\mathcal{N}) \cup X$ is satisfiable. But $\mathcal{N}$ does not satisfy $Th(\mathcal{N}) \cup X$. Hence, there are non-standard models of $Th(\mathcal{N})$ that have infinitely large numbers.

In particular, first-order logic is no help in obtaining structural characterizations.

The infinitary logic $L_{\omega_1\omega}$ is an extension of first-order logic that allows formulas to have infinite conjunctions and disjunctions.

Example:

$$\psi = \forall x \bigvee_{n\in\mathbb{N}} (x = \mathbf{n}) \wedge \forall x \exists y (y > x)$$

$\mathcal{N}$ is the unique structure that satisfies the discrete semiring part of PA and ψ. The finite conjunction of all these sentences is a *Scott sentence* for $\mathcal{N}$.

Theorem (Scott 1963)

For every countable structure $\mathcal{A}$ there is a sentence in the infinitary logic $L_{\omega_1\omega}$ – its Scott sentence – characterizing $\mathcal{A}$ up to isomorphism among countable structures.

Measure the complexity of a structure by the complexity of its Scott sentence.

1. A formula is $\Sigma^{\text{in}}_0 = \Pi^{\text{in}}_0$ if it is a finite quantifier free formula.
2. A formula is $\Sigma^{\text{in}}_\alpha$ for $\alpha > 0$, if it is of the form $\bigvee\!\!\!\bigvee_{i\in\omega} \exists \bar{x}_i \psi_i(\bar{x}_i)$ where all $\psi_i \in \Pi^{\text{in}}_{\beta_i}$ for $\beta_i < \alpha$.
3. A formula is Π^{in}_α for $\alpha > 0$, if it is of the form $\bigwedge\!\!\!\bigwedge_{i\in\omega} \forall \bar{x}_i \psi_i(\bar{x}_i)$ where all $\psi_i \in \Sigma^{\text{in}}_{\beta_i}$ for $\beta_i < \alpha$.
4. $L_{\omega_1\omega} = \bigcup_{\alpha<\omega_1} \Pi^{\text{in}}_\alpha$

- $\mathcal{N}$ has a Π^{in}_2 Scott sentence.
- Let p_n denote the (formal term) for the nth prime in PA and let $X \subseteq \omega$. Then

$$\varphi = \exists x \left(\bigwedge\!\!\!\bigwedge_{n\in X} \exists y (y \cdot p_n = x) \wedge \bigwedge\!\!\!\bigwedge_{n\notin X} \forall y (y \cdot p_n \neq x) \right)$$

is a Σ^{in}_3 formula and $\mathcal{A} \models \varphi$ iff X is in the Scott set of $\mathcal{A}$.

The proof of Scott's theorem heavily relies on the analysis of the α-back-and-forth relations for countable ordinals α. The most useful definition is due to Ash and Knight:

Definition

1. $(\mathcal{A},\bar{a}) \leq_0 (\mathcal{B},\bar{b})$ if all atomic fromulas true of $\bar{b}$ are true of $\bar{a}$ and vice versa.
2. For non-zero $\gamma < \omega_1$, $(\mathcal{A},\bar{a}) \leq_\gamma (\mathcal{B},\bar{b})$ if for all $\beta < \gamma$ and $\bar{d} \in B^{<\omega}$ there is $\bar{c} \in A^{<\omega}$ such that $(\mathcal{B},\bar{b}\bar{d}) \leq_\beta (\mathcal{A},\bar{a}\bar{c})$.

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In an attempt to measure structural complexity, various notions of ranks have been used.

E.g. $r(\mathcal{A})$ is the least α such that for all $\bar{a},\bar{b} \in A$ if $\bar{a} \leq_\alpha \bar{b}$, then $\bar{a} \leq_\beta \bar{b}$ for all $\beta > \alpha$.

Theorem (Montalbán 2015)

The following are equivalent for countable $\mathcal{A}$ and $\alpha < \omega_1$.

1. *Every automorphism orbit of $\mathcal{A}$ is $\Sigma^{\mathrm{in}}_{\alpha}$-definable without parameters.*
2. *$\mathcal{A}$ has a $\Pi^{\mathrm{in}}_{\alpha+1}$ Scott sentence.*
3. *$\mathcal{A}$ is uniformly $\mathbf{\Delta}^0_{\alpha}$-categorical. ($\exists\Phi\exists X\forall\mathcal{B}\cong\mathcal{C}\cong\mathcal{A}(\Phi^{X\oplus(\mathcal{C}\oplus\mathcal{B})^{(\alpha)}}:\mathcal{B}\cong\mathcal{C}$)*
4. *$Iso(\mathcal{A})$ is $\mathbf{\Pi}^0_{\alpha+1}$.*
5. *No tuple in $\mathcal{A}$ is α-free.*

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The standard model $\mathcal{N}$ has Scott rank 1 as it has a Π^{in}_2 Scott sentence.

Connection to $\leq_\alpha$

Theorem (Ash, Knight)

For two countable structures $\mathcal{A}$ the following are equivalent.

1. $(\mathcal{A}, \bar{a}) \leq_\alpha (\mathcal{B}, \bar{b})$.
2. *All $\Sigma_\alpha^{\mathrm{in}}$ sentences true of $\bar{b}$ in $\mathcal{B}$ are true of $\bar{a}$ in $\mathcal{A}$.*
3. *All Π_α^{in} sentences true of $\bar{a}$ in $\mathcal{A}$ are true of $\bar{b}$ in $\mathcal{B}$.*

In other words, $(\mathcal{A}, \bar{a}) \leq_\alpha (\mathcal{B}, \bar{b})$ iff $\Pi_\alpha^{\mathrm{in}}\text{-}tp^{\mathcal{A}}(\bar{a}) \subseteq \Pi_\alpha^{\mathrm{in}}\text{-}tp^{\mathcal{B}}(\bar{b})$.

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Definition

A tuple $\bar{a}$ in $\mathcal{A}$ is *α-free* if

$$\forall(\beta < \alpha)\forall \bar{b} \exists \bar{a}' \bar{b}' (\bar{a}\bar{b} \leq_\beta \bar{a}'\bar{b}' \wedge \bar{a} \not\leq_\alpha \bar{a}').$$

Definition (Makkai 1981)

The *Scott spectrum* of a theory T is the set

$$SS(T) = \{\alpha \in \omega_1 : \text{there is a countable model of } T \text{ with Scott rank } \alpha\}.$$

Here T might be a sentence in $L_{\omega_1 \omega}$.

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- Ash (1986) characterized back-and-forth relations of well-orderings. The following is a corollary: $SR(n) = 1$, $SR(\omega^\alpha) = 2\alpha$, $SR(\omega^\alpha + \omega^\alpha) = 2\alpha + 1$.
- $SS(LO) = \omega_1 - 0$
- $1 \in SS(PA)$

Throughout this talk $\mathcal{M}$ and $\mathcal{N}$ denote countable non-standard models of PA.

Recall that $\mathcal{M}$-finite sets can be coded by single elements, i.e., given $S \subseteq_{fin} M$ code it using $\sum_{s \in S} 2^s$. Thus finite strings $\bar{u} \in M^{<\omega}$ can be considered as the $\mathcal{M}$-finite set $\{\langle i, \bar{u}(i)\rangle : i < |\bar{u}|\}$.

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Let $Tr_{\Delta_1^0}$ be a truth predicate for bounded formulas and define the formal back-and-forth relations by induction on n:

$$\bar{u} \leq_0^a \bar{v} \Leftrightarrow \forall (x \leq a)(Tr_{\Delta_1^0}(x, \bar{u}) \to Tr_{\Delta_1^0}(x, \bar{v}))$$
$$\bar{u} \leq_{n+1}^a \bar{v} \Leftrightarrow \forall \bar{x} \exists \bar{y}\Big(|\bar{x}| \leq a \to (|\bar{y}| \leq a \wedge \bar{v}\bar{x} \leq_n^a \bar{u}\bar{y})\Big)$$

Proposition

The formal back-and-forth relations $\leq_n^x$ satisfy the following properties for all n:

1. $PA \vdash \forall \bar{u}, \bar{v}, a, b((a \leq b \wedge \bar{u} \leq_n^b \bar{v}) \rightarrow \bar{u} \leq_n^a \bar{v})$
2. $PA \vdash \forall \bar{u}, \bar{v}, a(\bar{u} \leq_{n+1}^a \bar{v} \rightarrow \bar{u} \leq_n^a \bar{v})$

Proposition

Let $\bar{a}, \bar{b} \in M$. Then $\bar{a} \leq_n \bar{b} \Leftrightarrow \forall (m \in \omega) \mathcal{M} \models \bar{a} \leq_n^{\dot{m}} \bar{b}$. Furthermore, if there is $c \in M - \mathbb{N}$ such that $\mathcal{M} \models \bar{a} \leq_n^c \bar{b}$, then $\bar{a} \leq_n \bar{b}$.

Lemma

For every $\bar{a}, \bar{b} \in M^{<\omega}$, $\bar{a} \leq_\omega \bar{b}$ *if and only if* $tp(\bar{a}) = tp(\bar{b})$.

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Recall that $\mathcal{M}$ is ***homogeneous*** if every partial elementary map $M \to M$ is extendible to an automorphism.

Lemma

If $\mathcal{M}$ is not homogeneous then $SR(\mathcal{M}) > \omega$.

Proposition

If $\mathcal{M}$ is homogeneous, then $SR(\mathcal{M}) \leq \omega + 1$.

Note that every completion T of PA has an atomic model. Take $\mathcal{M} \subseteq T$ and the subset of all Skolem terms without parameters. This is an elementary substructure and all types are isolated. By the least number principle this model is rigid and its automorphism orbits in $\mathcal{M}$ are singletons.

Theorem (Montalbán, R.)

If $\mathcal{M}$ is atomic, then $SR(\mathcal{M}) = \omega$.

Theorem (Montalbán, R.)

For any nonstandard model $\mathcal{M}$, $SR(\mathcal{M}) \geq \omega$. In particular $(1, \omega) \cap SS(PA) = \emptyset$. If $T \supseteq PA$ does not have a standard model, then $1 \notin SS(T)$.

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Definition (Harrison-Trainor, R. Miller, Montalbán 2018)

A structure $\mathcal{A} = (A, P_0^{\mathcal{A}}, \dots)$ is *infinitary interpretable* in $\mathcal{B}$ if there exists a $L_{\omega_1\omega}$ definable in $\mathcal{B}$ sequence of relations $(Dom_{\mathcal{A}}^{\mathcal{B}}, \sim, R_0, \dots)$ such that

1. $Dom_{\mathcal{A}}^{\mathcal{B}} \subseteq B^{<\omega}$,
2. $\sim$ is an equivalence relation on $Dom_{\mathcal{A}}^{\mathcal{B}}$,
3. $R_i \subseteq (B^{<\omega})^{a_{P_i}}$ is closed under $\sim$ on $Dom_{\mathcal{A}}^{\mathcal{B}}$,

and there exists a function $f_{\mathcal{B}}^{\mathcal{A}} : (Dom_{\mathcal{A}}^{\mathcal{B}}, R_0, \dots)/\sim \, \cong (A, P_0^{\mathcal{A}}, \dots)$, the *interpretation of $\mathcal{A}$ in $\mathcal{B}$*. If the formulas in the interpretation are $\Delta_\alpha^{\text{in}}$ then $\mathcal{A}$ is $\Delta_\alpha^{\text{in}}$ interpretable in $\mathcal{B}$.

Definition (Harrison-Trainor, R. Miller, Montalbán 2018)

Two structures $\mathcal{A}$ and $\mathcal{B}$ are *bi-interpretable* if there are infinitary interpretations of one in the other such that the compositions

$$f^{\mathcal{A}}_{\mathcal{B}} \circ \hat{f}^{\mathcal{B}}_{A} : Dom_{\mathcal{B}}^{Dom^{\mathcal{B}}_{\mathcal{A}}} \to \mathcal{B} \quad \text{and} \quad f^{\mathcal{B}}_{\mathcal{A}} \circ \hat{f}^{\mathcal{A}}_{\mathcal{B}} : Dom_{\mathcal{A}}^{Dom^{\mathcal{A}}_{\mathcal{B}}} \to \mathcal{A}$$

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Theorem (Harrison-Trainor, R. Miller, Montalbán 2018)

$\mathcal{A}$ and $\mathcal{B}$ are infinitary bi-interpretable iff their automorphism groups are Borel-measurably isomorphic.

Theorem (Gaifman 1976)

Let T be a completion of PA and $\mathcal{L}$ a linear order. Then there is a model $\mathcal{N}_{\mathcal{L}}$ of T such that $Aut(\mathcal{N}_{\mathcal{L}}) \cong Aut(\mathcal{L})$.

- A ***cut*** of a model $\mathcal{M}$ is a non-empty initial segment of $\mathcal{M}$ closed under successor.
- $\mathcal{N}$ is an ***end-extension*** of $\mathcal{M}$ if $\mathcal{M} \preccurlyeq \mathcal{N}$ and $\mathcal{M}$ is a cut of $\mathcal{N}$.
- $\mathcal{N}$ is a minimal extension of $\mathcal{M}$ if there is no $\mathcal{K}$ with $\mathcal{M} \prec \mathcal{K} \prec \mathcal{N}$.

Theorem (Gaifman 1976)

Let $\mathcal{M}$ be any model of PA, then $\mathcal{M}$ has a minimal end extension.

$\mathcal{L}$-canonical extension

The minimal end extension is obtained by taking $\mathcal{M}(a)$, the Skolem hull of $\mathcal{M}$ with a new element a having type $p(x)$ where

- $p(x)$ is *indiscernible*: for $I \subseteq M$ with every $i \in I$ having type $p(x)$ and ordered sequences $\bar{a}, \bar{b} \in I^{<\omega}$, $tp(\bar{a}) = tp(\bar{b})$,
- $p(x)$ is *unbounded*: there is no Skolem constant c such that $x \leq c \in p(x)$.

The version of Gaifman's theorem above is obtained by taking an $\mathcal{L}$-canonical extension for given $\mathcal{L}$ over the prime model $\mathcal{N}$, i.e., take an indiscernible, unbounded type $p(x)$, and construct the model

$$\mathcal{N}_{\mathcal{L}} = \bigcup_{l_1 \leq \cdots \leq l_{|l|} \in L^{<\omega}} \mathcal{N}(l_1)(l_2) \ldots (l_{|l|})$$

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Analysis of the construction shows that the elementary diagram of $\mathcal{N}_{\mathcal{L}}$ is Δ^{in}_1 interpretable in $\mathcal{L}$.

We still need to recover $\mathcal{L}$ from $\mathcal{N}_{\mathcal{L}}$ to obtain a bi-interpretation

Definition

Fix $\mathcal{M} \models PA$ and let $\mathcal{F}$ be the set of definable functions $f : M \to M$ for which $x \leq f(x) \leq f(y)$ whenever $x \leq y$. For any $a \in M$ let $gap(a)$ be the smallest set S with $a \in S$ and if $b \in S$, $f \in \mathcal{F}$, and $b \leq x \leq f(b)$ or $x \leq b \leq f(x)$, then $x \in S$.

Define $a =_g b$ as $a =_g b \Leftrightarrow a \in gap(b)$. The gap relation partitions $\mathcal{M}$ into intervals.

Theorem (Gaifman 1976)

- *If $a \in gap(b)$ and a, b both realize the same minimal type $p(x)$, then $a = b$.*
- *If $a \in gap(b)$ and $a \models p(x)$, then $a \in Scl(b)$.*
- *$\mathcal{N}_{\mathcal{L}}/{=_g}$ is order isomorphic to $1 + \mathcal{L}$.*

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$a \in Dom^{\mathcal{L}}_{\mathcal{N}_{\mathcal{L}}} \Leftrightarrow tp(a) = p(x)$	$a \sim b \Leftrightarrow a = b$	$a \leq b \Leftrightarrow a \leq^{\mathcal{N}_{\mathcal{L}}} b$
Π^{in}_{ω}	Δ^0_1	Δ^0_1

Properties of the interpretation

- The elementary diagram of $\mathcal{N}_{\mathcal{L}}$ is Δ_1^{in} interpretable in $\mathcal{L}$
- $\mathcal{L}$ is $\Delta_{\omega+1}^{\mathrm{in}}$ interpretable in $\mathcal{N}_{\mathcal{L}}$
- $\mathcal{L}$ and $\mathcal{N}_{\mathcal{L}}$ are $\Delta_{\omega+1}^{\mathrm{in}}$ bi-interpretable
- The interpretation is “asymmetric”

What could be the reason for that? It turns out we can interpret even more in $\mathcal{L}$!

Definition

Given a τ-structure $\mathcal{A}$ and a countable ordinal $\alpha > 0$ fix an injective enumeration $(\bar{a}_i)_{i\in\omega}$ of representatives of the α-back-and-forth equivalence classes in $\mathcal{A}$. The *canonical structural α-jump* $\mathcal{A}_{(\alpha)}$ of $\mathcal{A}$ is the structure in the vocabulary $\tau_{(\alpha)}$ obtained by adding to τ relation symbols R_i interpreted as

$$\bar{b} \in R_i^{\mathcal{A}_{(\alpha)}} \Leftrightarrow \bar{a}_i \leq_\alpha \bar{b}.$$

We will use the convention that $\mathcal{A}_{(0)} = \mathcal{A}$.

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Proposition

Let $\mathcal{A}$ be a τ-structure and φ be a Π^{in}_α τ-formula. Then there is a Σ^{in}_1 $\tau_{(\alpha)}$ formula ψ such that for all $\bar{a} \in A^{<\omega}$

$$(\mathcal{A}, \bar{a}) \models \varphi(\bar{a}) \Leftrightarrow (\mathcal{A}_{(\alpha)}, \bar{a}) \models \psi(\bar{a}).$$

Properties of the structural α-jump

Proposition

Let $\mathcal{A}$ be a structure and $\alpha, \beta < \omega_1$ where $\beta > 0$. Then
$(\mathcal{A}_{(\alpha)}, \bar{a}) \leq_\beta (\mathcal{A}_{(\alpha)}, \bar{b}) \Leftrightarrow (\mathcal{A}, \bar{a}) \leq_{\alpha+\beta} (\mathcal{A}, \bar{b})$.

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Corollary

For any structure $\mathcal{A}$ and non-zero $\alpha, \beta < \omega_1$, $SR(\mathcal{A}) = \alpha + \beta$ if and only if $SR(\mathcal{A}_{(\alpha)}) = \beta$.

Recall that two Δ_1^{in} bi-interpretable structures have the same Scott rank. So if $\mathcal{B}$ is Δ_1^{in} bi-interpretable with $\mathcal{A}_{(\alpha)}$, then $SR(\mathcal{A}) = \alpha + SR(\mathcal{B})$.

Let Γ be a set of formulas. Then Γ is $\Pi^{\mathrm{in}}_{\alpha}$-supported in $\mathcal{A}$ if there is a $\Pi^{\mathrm{in}}_{\alpha}$ formula φ such that

$$\mathcal{A} \models \exists \bar{x} \varphi(\bar{x}) \land \forall \bar{x} \left(\varphi(\bar{x}) \rightarrow \bigwedge\!\!\!\bigwedge_{\gamma \in \Gamma} \gamma(\bar{x}) \right).$$

Let Γ be a set of formulas. Then Γ is $\Pi^{\mathrm{in}}_{\alpha}$-supported in $\mathcal{A}$ if there is a $\Pi^{\mathrm{in}}_{\alpha}$ formula φ such that

$$\mathcal{A} \models \exists \bar{x} \varphi(\bar{x}) \wedge \forall \bar{x} \left(\varphi(\bar{x}) \rightarrow \bigwedge\!\!\!\bigwedge_{\gamma \in \Gamma} \gamma(\bar{x}) \right).$$

For $\alpha > 1$, there are uncountably many $\Pi^{\mathrm{in}}_{\alpha}$-types. The following might thus be a little bit surprising.

Proposition (Montalbán)

For every ordinal, every structure $\mathcal{A}$ and every tuple $\bar{a} \in A^{<\omega}$, $\Pi^{\mathrm{in}}_{\alpha}$-$tp^{\mathcal{A}}(\bar{a})$ is $\Pi^{\mathrm{in}}_{\alpha}$-supported in $\mathcal{A}$.

Corollary

For any structure $\mathcal{A}$ and non-zero ordinal α, $\mathcal{A}_{(\alpha)}$ is $\Delta^{\mathrm{in}}_{\alpha+1}$ interpretable in $\mathcal{A}$.

Corollary

For all countable ordinals α and β, the following are equivalent.

1. $\mathcal{A}_{(\gamma)}$ *is* Δ^{in}_1 *bi-interpretable with* $\mathcal{B}_{(\alpha)}$.
2. $\mathcal{A}$ *is infinitary bi-interpretable with* $\mathcal{B}$ *such that*
   - 2.1 *the interpretation of* $\mathcal{A}$ *in* $\mathcal{B}$ *and* $f^{\mathcal{A}}_{\mathcal{B}} \circ \tilde{f}^{\mathcal{B}}_{\mathcal{A}}$ *are* $\Delta^{\mathrm{in}}_{\alpha+1}$ *in* $\mathcal{B}$,
   - 2.2 *the interpretation of* $\mathcal{B}$ *in* $\mathcal{A}$ *and* $f^{\mathcal{B}}_{\mathcal{A}} \circ \tilde{f}^{\mathcal{A}}_{\mathcal{B}}$ *are* $\Delta^{\mathrm{in}}_{\gamma+1}$ *in* $\mathcal{A}$,
   - 2.3 *for every* $\bar{a} \in Dom^{\mathcal{B}}_{\mathcal{A}}$, $\{\bar{c} : (\mathcal{A}^{\mathcal{B}}, \bar{c}) \models \Pi^{\mathrm{in}}_{\gamma}\text{-}tp^{\mathcal{A}^{\mathcal{B}}}(\bar{a})\}$ *is* $\Delta^{\mathrm{in}}_{\alpha+1}$ *definable in* $\mathcal{B}$,
   - 2.4 *for every* $\bar{b} \in Dom^{\mathcal{A}}_{\mathcal{B}}$, $\{\bar{c} : (\mathcal{B}^{\mathcal{A}}, \bar{c}) \models \Pi^{\mathrm{in}}_{\alpha}\text{-}tp^{\mathcal{B}^{\mathcal{A}}}(\bar{b})\}$ *is* $\Delta^{\mathrm{in}}_{\gamma+1}$ *definable in* $\mathcal{A}$.

Recall that $\mathcal{N}_{\mathcal{L}}$ is Δ^{in}_1 interpretable in $\mathcal{L}$ and $\mathcal{L}$ is $\Delta^{\mathrm{in}}_{\omega+1}$ interpretable in $\mathcal{N}_{\mathcal{L}}$. Hence, taking $\mathcal{A} = \mathcal{L}$ and $\mathcal{B} = \mathcal{N}_{\mathcal{L}}$, 2.1, 2.2 are satisfied for $\alpha = \omega$, $\gamma = 0$. It remains to show that the elements satisfying a fixed $\Pi^{\mathrm{in}}_{\omega}$-type in $\mathcal{N}_{\mathcal{L}}$ are both Δ^{in}_1 definable in $\mathcal{L}$.

Gaifman's reduction shows that the elementary diagram of $\mathcal{N}_{\mathcal{L}}$ is Δ^{in}_1 interpretable in $\mathcal{L}$. By our result we get that $\{\bar{b} \vDash \Pi^{\text{in}}_{\omega}\text{-}tp(\bar{a})\} = \{\bar{b} \vDash tp(\bar{a})\}$. Clearly, for given $\bar{a} \in Dom^{\mathcal{L}}_{\mathcal{N}_{\mathcal{L}}}$ the sets $\{\bar{b} \vDash tp(\bar{a})\}$ is Π^{in}_1 definable in $\mathcal{L}$.

Gaifman's reduction shows that the elementary diagram of $\mathcal{N}_{\mathcal{L}}$ is Δ^{in}_1 interpretable in $\mathcal{L}$. By our result we get that $\{\bar{b} \models \Pi^{\text{in}}_\omega\text{-}tp(\bar{a})\} = \{\bar{b} \models tp(\bar{a})\}$. Clearly, for given $\bar{a} \in Dom^{\mathcal{L}}_{\mathcal{N}_{\mathcal{L}}}$ the sets $\{\bar{b} \models tp(\bar{a})\}$ is Π^{in}_1 definable in $\mathcal{L}$.

To show that it is also Π^{in}_1 definable notice that the following claim holds.

Lemma

Let s be a Skolem term and $a = s(l_1, \ldots, l_n)$ where $l_1 < \cdots < l_n \in L$. If $b = s(k_1, \ldots, k_n)$ for some $k_1 < \cdots < k_n \in L$ then $b \models tp(a)$.

Thus every set $\{\bar{b} \models tp(\bar{a})\}$ is the union of Skolem terms with parameters ordered $\mathcal{L}$-tuples. Let $(s_i)_{i\in\omega}$ be a listing of these Skolem terms for $tp(\bar{a})$. We get that the set is thus Π^{in}_1 definable.

Hence, $\mathcal{L}$ is Δ^{in}_1 bi-interpretable with $\mathcal{N}_{\mathcal{L}(\alpha)}$ and $SR(\mathcal{N}_{\mathcal{L}}) = \omega + S(\mathcal{L})$.

Theorem (Montalbán, R.)

1. $SS(PA) = 1 \cup \{\alpha : \omega \leq \alpha \leq \omega_1\}$
2. *If $\mathcal{M}$ is non-homogeneous, then $SR(\mathcal{M}) \geq \omega + 1$.*
3. *If $\mathcal{M}$ is non-standard atomic , then $SR(\mathcal{M}) = \omega$.*
4. *If $\mathcal{M}$ is non-standard homogeneous, then $SR(\mathcal{M}) \in [\omega, \omega + 1]$.*

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Thank you!