The degrees of categoricity above $\mathbf{0}''$

joint work with Barbara Csima

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A structure $\mathcal{A}$ in vocabulary τ is computable if there is an algorithm that computes $R_i^{\mathcal{A}}, f_i^{\mathcal{A}}, c_i^{\mathcal{A}}$ for all $R_, f_i, c_i \in \tau$.

Question: Given a computable structure $\mathcal{A}$, what is the least Turing degree that computes an isomorphism between all computable isomorphic copies of $\mathcal{A}$?—the ***degree of categoricity*** of $\mathcal{A}$.

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Example 1: Consider the standard model of arithmetic $\mathcal{N}$ and $\mathcal{A} \cong \mathcal{B} \cong \mathcal{N}$. We can compute an isomorphism between $\mathcal{A}$ and $\mathcal{B}$ by $\mathbf{n}^{\mathcal{A}} \mapsto \mathbf{n}^{\mathcal{B}}$ ($\mathbf{n}^{\mathcal{A}}$ is the value of the term representing n in $\mathcal{A}$.)

$\implies dgcat(\mathcal{A}) = \mathbf{0}$.

Example 2: $dgcat(\omega) = \mathbf{0}'$

(1) $\mathcal{G} = 0 \leq 1 \leq 2 \leq 3 \leq \ldots$

(2) $\mathcal{B}$ is constructed using a computable 1–1 enumeration $k_0, k_1, \ldots$ of $\emptyset'$.

$$\mathcal{B} = \quad 0 \; < \; 2 \; < \cdots < 2n < \; 2n+2 \; < \cdots$$

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(3) $Succ^{\mathcal{B}} \geq_T K$

(4) $(\forall f : \mathcal{G} \to \mathcal{B})\; f \geq_T \emptyset'$

(5) $\mathbf{0}'$ computes isomorphisms between any two computable copies of ω.

Example 3:

(1) $\mathcal{B} = \mathcal{H} \cong \omega_1^{\mathrm{CK}} + \omega_1^{\mathrm{CK}} \cdot \eta$, a computable copy of the Harrison order without HYP descending sequence

(2) $\mathcal{G} = \langle 0, \mathcal{H} \rangle + \cdots + \langle 2, \mathcal{H} \rangle + \langle 1, \mathcal{H} \rangle \cong \mathcal{H} + \sum_{i \in \omega_*} \mathcal{H} \cong \mathcal{H}$

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(3) If $dgcat(\mathcal{H}) \in \mathrm{HYP}$, then there is $f : \mathcal{G} \to \mathcal{B} \in \mathrm{HYP}$ and $f(\langle 1, 0 \rangle), f(\langle 2, 0 \rangle), \ldots$ is a HYP descending sequence in $\mathcal{B}$.

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$\Longrightarrow$ $\mathcal{H}$ does not have HYP degree of categoricity.

Theorem (Csima, Franklin, Shore ’13)
Every degree of categoricity is hyperarithmetic.

Thus, $\mathcal{H}$ does not have degree of categoricity.

- Fröhlich and Shepherdson '56 and Malt'sev '62: Computable field with non-computable transcendence basis.
- What is the least α, such that a structure $\mathbf{0}^{(\alpha)}$ computes isomorphisms between all isomorphic copies of $\mathcal{A}$?—$\mathbf{0}^{(\alpha)}$*-computable categoricity*
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Question 1: What degrees can be degrees of categoricity? Classify the degrees of categoricity.

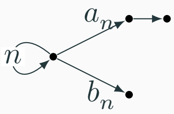
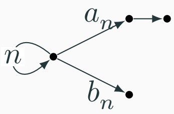
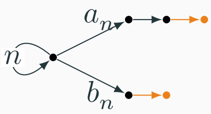
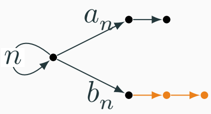
All the examples we considered had a good copy $\mathcal{G}$ and a bad copy $\mathcal{B}$ such that the isomorphisms between $\mathcal{G}$ and $\mathcal{B}$ witness the minimality of its degree of categoricity.

Definition
A degree of categoricity $\mathbf{d}$ is *strong* if there is $\mathcal{A}$ with $dgcat(\mathcal{A}) = \mathbf{d}$ and copies $\mathcal{G}$ and $\mathcal{B}$ such that for every isomorphism $f : \mathcal{G} \to \mathcal{B}$ $f \geq_T \mathbf{d}$.

Question 2: Is every degree of categoricity strong?

Fix c.e. $D \subseteq \omega$. We will construct two copies $\mathcal{G}$ and $\mathcal{B}$ of a graph.

They are the disjoint unions of the following connected components for all $n \in \omega$.

	$\mathcal{G}$	$\mathcal{B}$	
$n \notin D$			$a_n \mapsto a_n$ $b_n \mapsto b_n$
$n \searrow D$			$a_n \mapsto b_n$ $b_n \mapsto a_n$

Proposition

Every c.e. degree is a degree of categoricity.

The following are degrees of categoricity:

- Every degree d-c.e. in and above $\mathbf{0}^{(n)}$, $\mathbf{0}^{(\omega)}$ (Fokina, Kalimullin, Miller ’10)

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- Every Δ_2^0 degree is a degree of categoricity. (Csima, Ng '21):

All of these examples use similar coding ideas to the one for c.e. degrees. The codings get more and more complicated and are then combined with Marker extensions (Pairs of structures).

Definition
A degree $\mathbf{d}$ is ***treeable*** if there exists a computable tree $T \subseteq \omega^{<\omega}$ and $f \in \mathbf{d}$ such that $f \in [T]$ and $(\forall g \in [T]) f \leq_T g$.

Theorem (Csima, R. '22)
Every degree of categoricity is treeable.

Proof sketch.
Let $I(\mathcal{A}) = \{e : \varphi_e \cong \mathcal{A}\}$. We have that f computes an isomorphism between every computable copy of $\mathcal{A}$ iff $\forall j(\exists g \leq_T f)(j \in I(\mathcal{A}) \to g : \varphi_e \cong \varphi_j)$.

If $\mathbf{d}$ is the degree of categoricity of $\mathcal{A}$, then $\mathbf{d} \in \mathrm{HYP}$. $I(\mathcal{A})$ is arithmetical in $\mathbf{d}$ and thus $I(\mathcal{A}) \in \mathrm{HYP}$. Hence, the above formula is Σ^1_1 and thus there is a computable tree T such that its paths are in degree-preserving bijection with solutions to the above formula. T witnesses treeability of $\mathbf{d}$. $\square$

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Question: Is every treeable degree a degree of categoricity?

Theorem (Turetsky '20)

(1) *There is a computable structure* $\mathcal{A}_1$ *that has degree of categoricity* $\mathbf{0}$ *but high Scott rank.*

(2) *There is a computable structure* $\mathcal{A}_2$ *without degree of categoricity and computable dimension 2.*

A structure $\mathcal{A}$ has ***computable dimension*** $n \in (\omega \cup \{\omega\})$ if it has n computable copies up to computable isomorphism.

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$\mathcal{A}_2$ is obtained from $\mathcal{A}_1$ by a simple trick.

Idea to obtain $\mathcal{A}_1$:

(1) Given a computable tree T build a computable structure $\mathcal{A}$ such that $aut(\mathcal{A}) - id \equiv_w [T]$

(2) Force $\mathcal{A}$ to have degree of categoricity $\mathbf{0}$

(3) Take T such that $[T] \cap HYP = \emptyset$

Getting rid of $0''$

Turetsky produces:

1. $\mathcal{A}_1$ such that $aut(\mathcal{A}_1) - id \equiv_w [Q]$ and $\{f \oplus \emptyset'' : f \in [Q]\} \equiv_w \{f \oplus \emptyset'' : f \in [T]\}$
2. $\mathcal{A}_2 \cong \mathcal{G} \cong \mathcal{B}$ such that $\{f : (f : \mathcal{G} \cong \mathcal{B})\} \equiv_w [Q]$

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If we were able to get rid of the $\mathbf{0}''$, then we would get that every treeable degree is a degree of categoricity using the structures $\mathcal{A}_2$.

This seems to be difficult:

- All natural structures have computable dimension 1 and ω.
- Producing something with finite computable dimension seems to rely on infinite injury.
- (Goncharov) If a structure has degree of categoricity less than $\mathbf{0}'$, then it has computable dimension 1 or ω.

We can add the following to Turetsky's construction:

2. Force $\mathcal{A}$ to have degree of categoricity $\mathbf{0}$ (This forces $\mathcal{A}$ to code Q)

2a. For every $f \in [Q]$ $f \geq_T \emptyset''$.

Then $\{f : f : \mathcal{G} \cong \mathcal{B}\} \equiv_w [Q] \equiv_w \{f \oplus \emptyset'' : f \in [Q]\} \equiv_w \{f \oplus \emptyset'' : f \in [T]\}$.

Theorem (Csima, R.)
Every treeable degree $\mathbf{d}$ *such that* $\mathbf{d} \geq_T \mathbf{0}''$ *is the degree of categoricity of a structure.*

Theorem (Csima, R.)
Every degree of categoricity above $\mathbf{0}''$ *is strong.*

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Every degree of categoricity above $\mathbf{0}''$ is strong.

Congrats to this abstract characterization, is this actually useful?

$f \in \omega^\omega$ is a (Π^0_1) function singleton if there is a computable tree T with $[T] = \{f\}$.

Observation: The degree of every function singleton above $\mathbf{0}''$ is the degree of categoricity of a rigid structure of comp. dimension 2.

The following degrees are degrees of function singletons:

- (folklore) For all computable ordinals α, $\mathbf{0}^{(\alpha)}$ is the degree of a function singleton.
- (Jockusch, MacLaughlin '69) If $\mathbf{d}$ contains a function singleton, then so does every $\mathbf{c}$ with $\mathbf{d} \leq \mathbf{c} \leq \mathbf{d}'$.
- (Harrington '76) There is a non-arithmetical function singleton h such that $h^{(n)} \not\geq \mathbf{0}^{(\omega)}$ for all $n \in \omega$.

Corollary

(1) *For every computable* $\alpha \geq 2$*, every degree* $\mathbf{d} \in [\mathbf{0}^{(\alpha)}, \mathbf{0}^{(\alpha+1)}]$ *is a degree of categoricity.*

(2) *There is a degree* $\mathbf{d}$ *such that for every* $n \in \omega$*,* $\mathbf{c} \in [\mathbf{d}^{(n)}, \mathbf{d}^{(n+1)}]$ *is a non-arithmetic degree of categoricity.*

Proposition (Csima, R.)

Every degree $\mathbf{d} \in [\mathbf{0}', \mathbf{0}'']$ *is a degree of categoricity.*

Eliminating $\mathbf{0}''$ could be possible but will require new techniques:

1. (Goncharov) If a structure has degree of categoricity less than $\mathbf{0}'$, then it has computable dimension 1 or ω.
2. (Bazhenov, Yamaleev) There is a d-c.e. degree that is not the degree of categoricity of a rigid structure.

Theorem (Csima, R.)
There is a degree $\mathbf{d} \in (\mathbf{0}', \mathbf{0}'')$ that is not the degree of categoricity of a rigid structure.

We do not even know whether every function singleton is the degree of categoricity of a structure.

A degree $\mathbf{d}$ is *low for isomorphism* if whenever $\mathbf{d} \geq_T f : \mathcal{A}_1 \cong \mathcal{B}_1$ for $\mathcal{A}_1, \mathcal{B}_1$ computable, then they are computably isomorphic. (Franklin, Solomon '14)

A degree $\mathbf{d}$ is *low for paths* through Baire space if whenever $\mathbf{d} \geq_T f \in [T]$ for T in ω^ω computable, then T has a computable path.

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Theorem (Franklin, Turetsky '19)
A degree $\mathbf{d}$ is low for isomorphism if and only if it is low for paths.

All known($\star$) examples of hyperarithmetic degrees that are not degrees of categoricity are low for isomorphism. Thus, no known examples can be used to separate treeable degrees from degrees of categoricity.

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