

Positive enumerable functors

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Abstract. We study reductions well suited to compare structures and classes of structures with respect to properties based on enumeration reducibility. We introduce the notion of a positive enumerable functor and study the relationship with established reductions based on functors and alternative definitions.

1 Introduction

In this article we study notions of reductions that let us compare classes of structures with respect to their computability theoretic properties. Computability theoretic reductions between classes of structures can be formalized using effective versions of the category theoretic notion of a functor. While computable functors have already been used in the 80's by Goncharov [Gon80], the formal investigation of this notion was only started recently after R. Miller, Poonen, Schoutens, and Shlapentokh [Mil+18] explicitly used a computable functor to obtain a reduction from the class of graphs to the class of fields. Their result shows that fields are universal with respect to many properties studied in computable structure theory.

In [Ros17] the third author studied effective versions of functors based on enumeration reducibility and their relation to notions of interpretability. There, it was shown that the existence of a computable functor implies the existence of an enumerable functor effectively isomorphic to it. In that article there also appeared an unfortunately incorrect claim that enumerable functors are equivalent to a variation of effective interpretability, a notion equivalent to computable functors [Har+17]. Indeed, it was later shown in Rossegger's thesis [Ros19], that the existence of a computable functor implies the existence of an enumerable functor and thus enumerable functors are equivalent to the original notion. Hence, enumerable functors are equivalent to the original version of effective interpretability. In this paper we provide a simple proof of the latter result. It is not very surprising that enumerable and computable functors are equivalent, as the enumeration operators witnessing the effectiveness of an enumerable functor are given access to the atomic diagrams of structures, which are total sets.

The main objective of this article is the study of positive enumerable functors, an effectivization of functors that grants the involved enumeration operators

access to the positive diagrams of structures instead of their atomic diagrams. While computable functors are well suited to compare structures with respect to properties related to relative computability and the Turing degrees, positive enumerable functors provide the right framework to compare structures with respect to their enumerations and properties related to the enumeration degrees.

The paper is organized as follows. In Section 2, we first show that computable functors and enumerable functors are equivalent, and then begin the study of positive enumerable functors and reductions based on them. We show that reductions by positive enumerable bi-transformations preserve enumeration degree spectra, a generalization of degree spectra considering all enumerations of a structure introduced by Soskov [Sos04]. We then exhibit an example consisting of two structures which are computably bi-transformable but whose enumeration degree spectra are different. This implies that positive enumerable functors and computable functors are independent notions. Towards the end of the section we compare different possible definitions of positive enumerable functors and extend our results to reductions between arbitrary classes of structures.

2 Computable and enumerable functors

In this article we assume that our structures are in a relational language $(R_i)_{i \in \omega}$ where each R_i has arity a_i and the map $i \mapsto a_i$ is computable. We furthermore only consider countable structures with universe ω . We view classes of structures as categories where the objects are structures in a given language \mathcal{L} and the morphisms are isomorphisms between them. Recall that a functor $F : \mathfrak{C} \rightarrow \mathfrak{D}$ maps structures from \mathfrak{C} to structures in \mathfrak{D} and maps isomorphisms $f : \mathcal{A} \rightarrow \mathcal{B}$ to $F(f) : F(\mathcal{A}) \rightarrow F(\mathcal{B})$ preserving composition and identity.

The smallest classes we consider are isomorphism classes of a single structure \mathcal{A} ,

$$Iso(\mathcal{A}) = \{\mathcal{B} : \mathcal{B} \cong \mathcal{A}\}.$$

We will often talk about a functor from \mathcal{A} to \mathcal{B} , $F : \mathcal{A} \rightarrow \mathcal{B}$ when we mean a functor $F : Iso(\mathcal{A}) \rightarrow Iso(\mathcal{B})$. Depending on the properties that we want our functor to preserve we may use different effectivizations, but they will all be of the following form. Generally, an effectivization of a functor $F : \mathfrak{C} \rightarrow \mathfrak{D}$ will consist of a pair of operators (Φ, Φ_*) and a suitable coding C such that

1. for all $\mathcal{A} \in \mathfrak{C}$, $\Phi(C(\mathcal{A})) = C(F(\mathcal{A}))$,
2. for all $\mathcal{A}, \mathcal{B} \in \mathfrak{C}$ and $f \in Hom(\mathcal{A}, \mathcal{B})$, $\Phi(C(\mathcal{A}), C(f), C(\mathcal{B})) = C(F(f))$.

In this article the operators will either be enumeration or Turing operators. If the coding is clear from context we will omit the coding function, i.e., we write $\Phi(\mathcal{A})$ instead of $\Phi(C(\mathcal{A}))$. The most common coding in computable structure theory is the following.

Definition 1. Let \mathcal{A} be a structure in relational language $(R_i)_{i \in \omega}$. Then the atomic diagram $D(\mathcal{A})$ of \mathcal{A} is the set

$$\bigoplus_{i \in \omega} R_i^{\mathcal{A}} \oplus \bigoplus_{i \in \omega} \neg R_i^{\mathcal{A}}.$$

In the literature one can often find different definitions of the atomic diagram. It is easy to show that all of these notions are Turing and enumeration equivalent. The reason why we chose this definition is that it is conceptually easier to define the positive diagram and deal with enumerations of structures like this. We are now ready to define various effectivizations of functors.

Definition 2 ([Mil+18],[Har+17]). A functor $F : \mathfrak{C} \rightarrow \mathfrak{D}$ is computable if there is a pair of Turing operators (Φ, Φ_*) such that for all $\mathcal{A}, \mathcal{B} \in \mathfrak{C}$

1. $\Phi^{D(\mathcal{A})} = D(F(\mathcal{A}))$,
2. for all $f \in \text{Hom}(\mathcal{A}, \mathcal{B})$, $\Phi^{D(\mathcal{A}) \oplus \text{Graph}(f) \oplus D(\mathcal{B})} = F(f)$.

Definition 3. A functor $F : \mathfrak{C} \rightarrow \mathfrak{D}$ is enumerable if there is a pair (Ψ, Ψ_*) where Ψ and Ψ_* are enumeration operators such that for all $\mathcal{A}, \mathcal{B} \in \mathfrak{C}$

1. $\Psi^{D(\mathcal{A})} = D(F(\mathcal{A}))$,
2. for all $f \in \text{hom}(\mathcal{A}, \mathcal{B})$, $\Psi_*^{D(\mathcal{A}) \oplus \text{Graph}(f) \oplus D(\mathcal{B})} = \text{Graph}(F(f))$.

In [Ros17] enumerable functors were defined differently, using a Turing operator instead of an enumeration operator for the homomorphisms. The definition was as follows.

Definition 4 ([Ros17]). A functor $F : \mathfrak{C} \rightarrow \mathfrak{D}$ is \star -enumerable if there is a pair (Ψ, Φ_*) where Ψ is an enumeration operator and Φ_* is a Turing operator such that for all $\mathcal{A}, \mathcal{B} \in \mathfrak{C}$

1. $\Psi^{D(\mathcal{A})} = D(F(\mathcal{A}))$,
2. for all $f \in \text{hom}(\mathcal{A}, \mathcal{B})$, $\Phi_*^{D(\mathcal{A}) \oplus \text{Graph}(f) \oplus D(\mathcal{B})} = \text{Graph}(F(f))$.

It turns out that the two definitions are equivalent and we will thus stick with Definition 3 which seems to be more natural.

Proposition 5. A functor $F : \mathfrak{A} \rightarrow \mathfrak{B}$ is enumerable if and only if it is \star -enumerable.

Proof. Say we have an enumerable functor given by (Ψ, Ψ_*) and an isomorphism $f : \tilde{\mathcal{A}} \rightarrow \hat{\mathcal{A}}$ for $\tilde{\mathcal{A}} \cong \hat{\mathcal{A}} \cong \mathcal{A}$. We can compute the isomorphism $F(f)$ by enumerating $\text{Graph}(F(f))$ using $\Psi_*^{\tilde{\mathcal{A}} \oplus f \oplus \hat{\mathcal{A}}}$. For every x we are guaranteed to enumerate $(x, y) \in \text{Graph}(F(f))$ for some y as the domain of \mathcal{A} is ω . This is uniform in $\tilde{\mathcal{A}}$, f and $\hat{\mathcal{A}}$. Thus there is a Turing operator Φ_* such that (Ψ, Φ_*) witnesses that F is \star -enumerable.

Now, say F is \star -enumerable as witnessed by (Ψ, Φ_*) . For every σ, x, y with $\Phi_*^\sigma(x) \downarrow = y$ such that σ can be split into $\sigma_0 \oplus \sigma_1 \oplus \sigma_2$ where σ_0, σ_2 are partial characteristic functions of finite structures in a finite sublanguage L of the language of \mathcal{A} and σ_1 is the partial graph of a function, consider the set

$$\begin{aligned} X_\sigma^{x,y} = \{ & (B \oplus \text{Graph}(\tau) \oplus C, \langle x, y \rangle) : B, C \text{ are atomic diagrams of finite} \\ & L\text{-structures, } B \text{ compatible with } \sigma_0, C \text{ compatible with } \sigma_2, \\ & \sigma_1(u, v) = 1 \rightarrow \tau(u) = v, \text{ and } \sigma_1(u, v) = 0 \rightarrow \tau(u) = z \\ & \text{where } z \notin \text{range}(\sigma_1)\}. \end{aligned}$$

We can now define our enumeration operator as $\Psi_\star = \bigcup_{x,y,\sigma:\Phi_*^\sigma(x)\downarrow=y} X_\sigma^{x,y}$. Given an enumeration of Φ_* we can produce an enumeration of Ψ_\star , so Ψ_\star is c.e. It remains to show that $\Psi_\star^{\hat{\mathcal{A}} \oplus f \oplus \hat{\mathcal{A}}} = \Phi_*^{\hat{\mathcal{A}} \oplus f \oplus \hat{\mathcal{A}}}$.

Say $\Phi_*^{\hat{\mathcal{A}} \oplus f \oplus \hat{\mathcal{A}}}(x) = y$. Then there is $\sigma \preceq \tilde{\mathcal{A}} \oplus f \oplus \hat{\mathcal{A}}$ such that $(\sigma, x, y) \in \Phi_*$ and thus by the construction of X_σ there is $B \subseteq D(\tilde{\mathcal{A}})$, $C \subseteq D(\hat{\mathcal{A}})$ and $\text{Graph}(\tau) \subseteq \text{Graph}(f)$ such that $(B \oplus \text{Graph}(\tau) \oplus C, \langle x, y \rangle) \in X_\sigma$. Thus $\langle x, y \rangle \in \Psi_\star^{\hat{\mathcal{A}} \oplus f \oplus \hat{\mathcal{A}}}$.

On the other hand say $\langle x, y \rangle \in \Psi_\star^{\hat{\mathcal{A}} \oplus f \oplus \hat{\mathcal{A}}}$. Then, there is $(B \oplus \text{Graph}(\tau) \oplus C, \langle x, y \rangle) \in \Psi_\star$ with $B \subseteq \tilde{\mathcal{A}}$, $\text{Graph}(\tau) \subseteq \text{Graph}(f)$ and $C \subseteq \hat{\mathcal{A}}$. Furthermore, there is $\sigma \preceq \chi_{B \oplus \text{Graph}(\tau) \oplus C}$ such that $(\sigma, x, y) \in \Phi_*$. Thus $\Psi_\star^{\hat{\mathcal{A}} \oplus f \oplus \hat{\mathcal{A}}} = \text{Graph}(F(f))$ for any $\hat{\mathcal{A}} \cong \tilde{\mathcal{A}} \cong \mathcal{A}$ and $f : \tilde{\mathcal{A}} \cong \hat{\mathcal{A}}$ and hence F is enumerable. \square

In [Ros17] it was shown that the existence of an enumerable functor implies the existence of a computable functor and in [Ros19] the converse was shown. We give a simple proof of the latter.

Theorem 6. *If $F : \mathcal{A} \rightarrow \mathcal{B}$ is a computable functor, then it is enumerable.*

Proof. Given a computable functor F we will show that F is \star -enumerable. That F is then also enumerable follows from Proposition 5.

Let $D(L_{\mathcal{A}})$ be the collection of finite atomic diagrams in the language of \mathcal{A} . To every $p \in D(L_{\mathcal{A}})$ we associate a finite string α_p in the alphabet $\{0, 1, \uparrow\}$ so that if p specifies that R_i holds on elements coded by u , then we set that $\neg R_i$ does not hold on these elements. More formally, $\alpha_p(x) = 1$ if $x \in p$, $\alpha_p(x) = 0$ if $x = 2\langle i, u \rangle$ and $2\langle i, u \rangle + 1 \in p$ or $x = 2\langle i, u \rangle + 1$ and $2\langle i, u \rangle \in p$, and $\alpha_p(x) = \uparrow$ if x is less than the largest element of p and none of the other cases fits. We also associate a string $\tilde{\alpha}_p \in 2^{|\alpha_p|}$ with p where $\tilde{\alpha}_p(x) = 1$ if and only if $\alpha_p(x) = 1$ and $\tilde{\alpha}_p(x) = 0$ if and only if $\alpha_p(x) = 0$ or $\alpha_p(x) = \uparrow$.

Let the computability of F be witnessed by (Φ, Φ_*) . We build the enumeration operator Ψ as follows. For every $p \in D(L_{\mathcal{A}})$ and every x if $\Phi^{\tilde{\alpha}_p}(x) \downarrow = 1$ and every call to the oracle during the computation is on an element z such that $\alpha_p(z) \neq \uparrow$, then enumerate (p, x) into Ψ . This finishes the construction of Ψ .

Now, let $\hat{\mathcal{A}} \cong \mathcal{A}$. We have that $x \in \Psi^{\hat{\mathcal{A}}}(x)$ if and only if there exists $p \in D(L_{\mathcal{A}})$ such that $p \subseteq D(\hat{\mathcal{A}})$ and $(p, x) \in \Psi$. We further have that $(p, x) \in \Psi$ if and only if $\Phi^{\hat{\mathcal{A}}}(x) \downarrow = 1$ if and only if $\Phi^{\hat{\mathcal{A}}}(x) = 1$. Thus F is enumerable using (Ψ, Φ_*) . \square

Combining Theorem 6 with the results from [Ros17] we obtain that enumerable functors and computable functors defined using the atomic diagram of a structure as input are equivalent notions. This is not surprising. After all, the atomic diagram of a structure always has total enumeration degree and there is a canonical isomorphism between the total enumeration degrees and the Turing degrees. In order to make this equivalence precise we need another definition.

Definition 7 ([Har+17]). *A functor $F : \mathfrak{C} \rightarrow \mathfrak{D}$ is effectively isomorphic to a functor $G : \mathfrak{C} \rightarrow \mathfrak{D}$ if there is a Turing functional Λ such that for any $\mathcal{A} \in \mathfrak{C}$, $\Lambda^{\mathcal{A}} : F(\mathcal{A}) \rightarrow G(\mathcal{A})$ is an isomorphism. Moreover, for any morphism $h \in \text{Hom}(\mathcal{A}, \mathcal{B})$ in \mathfrak{C} , $\Lambda^{\mathcal{B}} \circ F(h) = G(h) \circ \Lambda^{\mathcal{A}}$. That is, the diagram below commutes.*

$$\begin{array}{ccc} F(\mathcal{A}) & \xrightarrow{\Lambda^{\mathcal{A}}} & G(\mathcal{A}) \\ F(h) \downarrow & & \downarrow G(h) \\ F(\mathcal{B}) & \xrightarrow{\Lambda^{\mathcal{B}}} & G(\mathcal{B}) \end{array}$$

The following is an immediate corollary of Theorem 6 and [Ros17, Theorem 2].

Theorem 8. *Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be a functor. Then F is computable if and only if there is an enumerable functor $G : \mathcal{A} \rightarrow \mathcal{B}$ effectively isomorphic to F .*

Definition 9 ([Har+17]). *Suppose $F : \mathfrak{C} \rightarrow \mathfrak{D}$, $G : \mathfrak{D} \rightarrow \mathfrak{C}$ are functors such that $G \circ F$ is effectively isomorphic to $\text{Id}_{\mathfrak{C}}$ via the Turing functional $\Lambda_{\mathfrak{C}}$ and $F \circ G$ is effectively isomorphic to $\text{Id}_{\mathfrak{D}}$ via the Turing functional $\Lambda_{\mathfrak{D}}$. If furthermore, for any $\mathcal{A} \in \mathfrak{C}$ and $\mathcal{B} \in \mathfrak{D}$, $\Lambda_{\mathfrak{D}}^{F(\mathcal{A})} = F(\Lambda_{\mathfrak{C}}^{\mathcal{A}}) : F(\mathcal{A}) \rightarrow F(G(F(\mathcal{A})))$ and $\Lambda_{\mathfrak{C}}^{G(\mathcal{B})} = G(\Lambda_{\mathfrak{D}}^{\mathcal{B}}) : G(\mathcal{B}) \rightarrow G(F(G(\mathcal{B})))$, then F and G are said to be pseudo inverses.*

Definition 10 ([Har+17]). *Two structures \mathcal{A} and \mathcal{B} are computably bi-transformable if there are computable pseudo-inverse functors $F : \mathcal{A} \rightarrow \mathcal{B}$ and $G : \mathcal{B} \rightarrow \mathcal{A}$.*

If the functors in Definition 10 are enumerable instead of computable then we say that \mathcal{A} and \mathcal{B} are *enumerably bi-transformable*. As an immediate corollary of Theorem 8 we obtain the following.

Corollary 11. *Two structures \mathcal{A} and \mathcal{B} are enumerably bi-transformable if and only if they are computably bi-transformable.*

3 Effectivizations using positive diagrams

We now turn our attention to the setting where we only have positive information about the structures. We follow Soskov [Sos04] in our definitions. See also the survey paper by Soskova and Soskova [SS17] on computable structure theory and enumeration degrees.

Definition 12. *Let \mathcal{A} be a structure in relational language $(R_i)_{i \in \omega}$. The positive diagram of \mathcal{A} , denoted by $P(\mathcal{A})$, is the set*

$$= \oplus \neq \oplus \bigoplus_{i \in \omega} R_i^{\mathcal{A}}.$$

We are interested in the degrees of enumerations of $P(\mathcal{A})$. To be more precise let f be an enumeration of ω and for $X \subseteq \omega^n$ let

$$f^{-1}(X) = \{\langle x_1, \dots, x_n \rangle : (f(x_1), \dots, f(x_n)) \in X\}.$$

Given \mathcal{A} let $f^{-1}(\mathcal{A}) = f^{-1}(=) \oplus f^{-1}(\neq) \oplus f^{-1}(R_0^{\mathcal{A}}) \oplus \dots$. Notice that if $f = id$, then f^{-1} is just the positive diagram of \mathcal{A} .

Definition 13. *The enumeration degree spectrum of \mathcal{A} is the set*

$$eSp(\mathcal{A}) = \{d_e(f^{-1}(\mathcal{A})) : f \text{ is an enumeration of } \omega\}.$$

If \mathbf{a} is the least element of $eSp(\mathcal{A})$, then \mathbf{a} is called the enumeration degree of \mathcal{A} .

In order to obtain a notion of reduction that preserves enumeration spectra we need an effectivization of functors where we use positive diagrams of structures as coding. It is clear that for computable functors this makes no difference as $P(\mathcal{A}) \equiv_T D(\mathcal{A})$. For enumerable functors it does make a difference. We also need to replace the Turing operators in the definition of pseudo inverses with enumeration operators. The new notions are as follows.

Definition 14. *A functor $F : \mathfrak{C} \rightarrow \mathfrak{D}$ is positive enumerable if there is a pair (Ψ, Ψ_*) where Ψ and Ψ_* are enumeration operators such that for all $\mathcal{A}, \mathcal{B} \in \mathfrak{C}$*

1. $\Psi^{P(\mathcal{A})} = P(F(\mathcal{A}))$,
2. for all $f \in \text{hom}(\mathcal{A}, \mathcal{B})$, $\Psi_*^{P(\mathcal{A}) \oplus \text{Graph}(f) \oplus P(\mathcal{B})} = \text{Graph}(F(f))$.

Definition 15. *A functor $F : \mathfrak{C} \rightarrow \mathfrak{D}$ is enumeration isomorphic to a functor $G : \mathfrak{C} \rightarrow \mathfrak{D}$ if there is an enumeration operator Λ such that for any $\mathcal{A} \in \mathfrak{C}$, $\Lambda^{P(\mathcal{A})} : F(\mathcal{A}) \rightarrow G(\mathcal{A})$ is an isomorphism. Moreover, for any morphism $h \in \text{Hom}(\mathcal{A}, \mathcal{B})$ in \mathfrak{C} , $\Lambda^{P(\mathcal{B})} \circ F(h) = G(h) \circ \Lambda^{P(\mathcal{A})}$.*

Definition 16. Suppose $F : \mathfrak{C} \rightarrow \mathfrak{D}$, $G : \mathfrak{D} \rightarrow \mathfrak{C}$ are functors such that $G \circ F$ is enumeration isomorphic to $Id_{\mathfrak{C}}$ via the enumeration operator $\Lambda_{\mathfrak{C}}$ and $F \circ G$ is enumeration isomorphic to $Id_{\mathfrak{D}}$ via the enumeration operator $\Lambda_{\mathfrak{D}}$. If, furthermore, for any $\mathcal{A} \in \mathfrak{C}$ and $\mathcal{B} \in \mathfrak{D}$, $\Lambda_{\mathfrak{D}}^{P(F(\mathcal{A}))} = F(\Lambda_{\mathfrak{C}}^{P(\mathcal{A})}) : F(\mathcal{A}) \rightarrow F(G(F(\mathcal{A})))$ and $\Lambda_{\mathfrak{C}}^{P(G(\mathcal{B}))} = G(\Lambda_{\mathfrak{D}}^{P(\mathcal{B})}) : G(\mathcal{B}) \rightarrow G(F(G(\mathcal{B})))$, then F and G are said to be enumeration pseudo inverses.

Definition 17. Two structures \mathcal{A} and \mathcal{B} are positive enumerably bi-transformable if there are positive enumerable enumeration pseudo-inverse functors $F : \mathcal{A} \rightarrow \mathcal{B}$ and $G : \mathcal{B} \rightarrow \mathcal{A}$.

Theorem 18. Let \mathcal{A} and \mathcal{B} be positive enumerably bi-transformable. Then $eSp(\mathcal{A}) = eSp(\mathcal{B})$.

Proof. Say \mathcal{A} and \mathcal{B} are positive enumerably bi-transformable by $F : \mathcal{A} \rightarrow \mathcal{B}$ and $G : \mathcal{B} \rightarrow \mathcal{A}$. Let f be an arbitrary enumeration of ω , then, viewing $f^{-1}(\mathcal{A})/f^{-1}(=)$ as a structure on ω by pulling back a canonical enumeration of the least elements in its $=$ -equivalence classes, we have that there is $\hat{\mathcal{A}} \cong \mathcal{A}$ such that $P(\hat{\mathcal{A}}) = f^{-1}(\mathcal{A})/f^{-1}(=)$ and $P(\hat{\mathcal{A}}) \leq_e f^{-1}(\mathcal{A})$. As F is positive enumerable we have that $f^{-1}(\mathcal{A}) \geq_e P(F(\hat{\mathcal{A}}))$. Furthermore, we shall see that $f^{-1}(F(\hat{\mathcal{A}})) \leq_e f^{-1}(\mathcal{A})$ and that $f^{-1}(\mathcal{A})/f^{-1}(=) = P(F(\hat{\mathcal{A}}))$. Given an enumeration of $f^{-1}(\mathcal{A})$ and an enumeration of $P(F(\hat{\mathcal{A}}))$, we may first order the equivalence classes of $f^{-1}(=)$ by their least elements and then, if $R_i(a_1, \dots, a_n) \in P(F(\hat{\mathcal{A}}))$ we enumerate $R_i(b_1, \dots, b_n)$ for all $b_1, \dots, b_n \in \omega$ such that b_j is in the a_j^{th} equivalence class of $f^{-1}(=)$. It is not hard to see that this gives an enumeration of a set X such that $f^{-1}(=) \oplus f^{-1}(\neq) \oplus X = f^{-1}(F(\hat{\mathcal{A}}))$, that $f^{-1}(F(\hat{\mathcal{A}}))/f^{-1}(=) = P(F(\hat{\mathcal{A}}))$, and since by construction $f^{-1}(F(\hat{\mathcal{A}})) \leq_e P(F(\hat{\mathcal{A}})) \oplus f^{-1}(\mathcal{A})$ we have $f^{-1}(F(\hat{\mathcal{A}})) \leq_e f^{-1}(\mathcal{A})$.

We can apply the same argument with G in place of F and $F(\hat{\mathcal{A}})$ in place of \mathcal{A} to get that $f^{-1}(G(F(\hat{\mathcal{A}})))/f^{-1}(=) = P(G(F(\hat{\mathcal{A}})))$ and

$$f^{-1}(G(F(\hat{\mathcal{A}}))) \leq_e f^{-1}(F(\hat{\mathcal{A}})) \leq_e f^{-1}(\mathcal{A}).$$

At last, recall that, as \mathcal{A} and \mathcal{B} are positive enumerably bi-transformable, there is an enumeration operator Ψ such that $\Psi^{P(G(F(\hat{\mathcal{A}})))}$ is the enumeration of the graph of an isomorphism $i : G(F(\hat{\mathcal{A}})) \cong \hat{\mathcal{A}}$. But then $(f \circ i)^{-1}(G(F(\hat{\mathcal{A}}))) = f^{-1}(\mathcal{A})$ and

$$f^{-1}(\mathcal{A}) \leq_e f^{-1}(G(F(\hat{\mathcal{A}}))) \leq_e f^{-1}(F(\hat{\mathcal{A}})) \leq_e f^{-1}(\mathcal{A}).$$

This shows that $eSp(\mathcal{A}) \subseteq eSp(\mathcal{B})$. The proof that $eSp(\mathcal{B}) \subseteq eSp(\mathcal{A})$ is analogous. \square

Proposition 19. There are computably bi-transformable structures \mathcal{A} and \mathcal{B} such that $eSp(\mathcal{A}) \neq eSp(\mathcal{B})$. In particular, \mathcal{A} and \mathcal{B} are not positive enumerably bi-transformable.

Proof. Let $\mathcal{A} = (\omega, \underline{0}, s, K)$ where s is the successor relation on ω , $\underline{0}$ the first element, and K the membership relation of the halting set. Assume $\mathcal{B} = (\omega, \underline{0}, s, \overline{K})$ is defined as \mathcal{A} except that $\overline{K}(x)$ if and only if $\neg K(x)$. There is a computable functor $F : \mathcal{A} \rightarrow \mathcal{B}$ taking $\hat{\mathcal{A}} = (\omega, \underline{0}^{\hat{\mathcal{A}}}, s^{\hat{\mathcal{A}}}, K^{\hat{\mathcal{A}}}) \cong \mathcal{A}$ to $F(\hat{\mathcal{A}}) = (\omega, \underline{0}^{\hat{\mathcal{A}}}, s^{\hat{\mathcal{A}}}, \neg K^{\hat{\mathcal{A}}})$ and acting as the identity on isomorphisms. Furthermore, F has a computable inverse and thus \mathcal{A} is computably bi-transformable to \mathcal{B} .

However, \mathcal{A} has enumeration degree $\mathbf{0}'_e$ and \mathcal{B} has enumeration degree $\overline{\mathbf{0}}'_e$. Thus there cannot be a positive enumerable functor from \mathcal{B} to \mathcal{A} . \square

The following shows that computable functors and positive enumerable functors are independent.

Proposition 20. *There are structures \mathcal{A} and \mathcal{B} such that \mathcal{A} is positive enumerably bi-transformable with \mathcal{B} but \mathcal{A} is not computably bi-transformable with \mathcal{B} .*

Proof. Let \mathcal{A} be as in Proposition 19, i.e., $\mathcal{A} = (\omega, \underline{0}, s, K)$ and $\mathcal{B} = (\omega, \underline{0}, s)$. Then it is not hard to see that \mathcal{A} is positive enumerably bi-transformable with \mathcal{B} . However, there can not be a computable functor from \mathcal{B} to \mathcal{A} as \mathcal{B} has Turing degree $\mathbf{0}$ and \mathcal{A} has Turing degree $\mathbf{0}'$. \square

We have seen in Proposition 5 that \star -enumerable functors and enumerable functors are equivalent. Positive enumerable functors also admit a different definition.

Definition 21. *A functor $F : \mathfrak{C} \rightarrow \mathfrak{D}$ is positive \star -enumerable if there is a pair (Ψ, Φ_*) where Ψ is an enumeration operator and Φ_* is a Turing operator such that for all $\mathcal{A}, \mathcal{B} \in \mathfrak{C}$*

1. $\Psi^{P(\mathcal{A})} = P(F(\mathcal{A}))$,
2. for all $f \in \text{hom}(\mathcal{A}, \mathcal{B})$, $\Phi_*^{P(\mathcal{A}) \oplus \text{Graph}(f) \oplus P(\mathcal{B})} = \text{Graph}(F(f))$.

Proposition 22. *Every positive enumerable functor is positive \star -enumerable.*

Proof. Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be given by (Ψ, Ψ_*) and let $f : \tilde{\mathcal{A}} \cong \hat{\mathcal{A}}$ for $\tilde{\mathcal{A}} \cong \hat{\mathcal{A}} \cong \mathcal{A}$. Now we can define a procedure computing $F(f)$ as follows. Given x , and $\tilde{\mathcal{A}} \oplus f \oplus \hat{\mathcal{A}}$ enumerate $\Psi_*^{\tilde{\mathcal{A}} \oplus f \oplus \hat{\mathcal{A}}}$ until $\langle x, y \rangle \searrow \Psi_*^{\tilde{\mathcal{A}} \oplus f \oplus \hat{\mathcal{A}}}$ for some y . This is uniform in $\tilde{\mathcal{A}} \oplus f \oplus \hat{\mathcal{A}}$ and thus there exists a Turing operator Φ_* with this behaviour. The pair (Ψ, Φ_*) then witnesses that F is \star -enumerable. \square

Theorem 23. *There is positive \star -enumerable functor that is not enumeration isomorphic to any positive enumerable functor.*

Proof. We will build two structures \mathcal{A} and \mathcal{B} such that there is a positive \star -enumerable functor $F : \mathcal{A} \rightarrow \mathcal{B}$ that is not positive enumerable. The structure \mathcal{A} is a graph constructed as follows. It has a vertex a with a loop connected to

a and a cycle of size n for every natural number n . If $n \in \emptyset'$ then there is an edge between a and one element of the n cycle, otherwise there is no such edge. Clearly, $\text{deg}_T(P(\mathcal{A})) = \mathbf{0}'$ and $P(\mathcal{A}) \not\geq_e \bar{\emptyset}'$.

The structure \mathcal{B} is a typical graph that witnesses that there is a structure with degree of categoricity $\mathbf{0}'$ (that is, $\mathbf{0}'$ is the least degree computing an isomorphism between any two computable copies of \mathcal{B}). Let us recap how we build two copies of \mathcal{B} , \mathcal{B}_1 and \mathcal{B}_2 such that $\mathbf{0}'$ is the least degree computing isomorphism between \mathcal{B}_1 and \mathcal{B}_2 . Both graphs consist of an infinite ray with a loop at its first element. Let v_i be the i^{th} element in the ray in \mathcal{B}_1 and \hat{v}_i be the i^{th} element in the ray in \mathcal{B}_2 . Now for every v_i there are two elements a_i and b_i with $v_i E a_i$ and $v_i E b_i$. Likewise for every \hat{v}_i there are two elements \hat{a}_i and \hat{b}_i with $\hat{v}_i E \hat{a}_i$ and $\hat{v}_i E \hat{b}_i$. Furthermore there are additional vertices s_i, \hat{s}_i with $a_i E s_i$ and $\hat{a}_i E \hat{s}_i$.

Take an enumeration of \emptyset' . If $i \searrow \emptyset'$, then add vertices $b_i E \cdot E$ and $\hat{s}_i E \cdot, \hat{b}_i E \cdot$. This finishes the construction of \mathcal{B} . It is not hard to see that there is a unique isomorphism $f : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ and that $\text{deg}(f) = \mathbf{0}'$ and $\text{Graph}(f) \geq_e \bar{\emptyset}'$.

We now construct the functor F . Given an enumeration of $P(\hat{\mathcal{A}})$ for $\hat{\mathcal{A}} \cong \mathcal{A}$ we wait until we see the cycle containing 0 (any natural number would work). If it is of even length, or 0 is the special vertex a , we let $F(\hat{\mathcal{A}}) = \mathcal{B}_1$ and if it is of odd length we let $F(\hat{\mathcal{A}}) = \mathcal{B}_2$. Clearly given any enumeration of a copy of \mathcal{A} this procedure produces an enumeration of a copy of \mathcal{B} .

As \mathcal{B} is rigid we just let $F(f : \hat{\mathcal{A}} \rightarrow \tilde{\mathcal{A}}) = g : F(\hat{\mathcal{A}}) \rightarrow F(\tilde{\mathcal{A}})$ where g is the unique isomorphism between $F(\hat{\mathcal{A}})$ and $F(\tilde{\mathcal{A}})$. Note that there is a Turing operator Θ such that $\Theta^{P(\hat{\mathcal{A}})} = \emptyset'$ for any $\hat{\mathcal{A}} \cong \mathcal{A}$ and that the isomorphism between $F(\hat{\mathcal{A}})$ and $F(\tilde{\mathcal{A}})$ can be computed uniformly from $P(F(\hat{\mathcal{A}})) \oplus P(F(\tilde{\mathcal{A}})) \oplus \emptyset'$. Thus, there is an operator Φ_* witnessing that F is positive \star -enumerable.

To see that F is not positive enumerable consider two copies $\hat{\mathcal{A}}$ and $\tilde{\mathcal{A}}$ of \mathcal{A} with $\text{deg}_e(P(\hat{\mathcal{A}})) = \text{deg}_e(P(\tilde{\mathcal{A}})) = \mathbf{0}'_e$ such that 0 is part of an even cycle in $\hat{\mathcal{A}}$ and part of an odd cycle in $\tilde{\mathcal{A}}$. Notice that there is $f : \hat{\mathcal{A}} \rightarrow \tilde{\mathcal{A}}$ such that $P(\hat{\mathcal{A}}) \oplus P(\tilde{\mathcal{A}}) \geq_e P(\hat{\mathcal{A}}) \oplus \text{Graph}(f : \hat{\mathcal{A}} \rightarrow \tilde{\mathcal{A}}) \oplus P(\tilde{\mathcal{A}})$, and also that $P(\hat{\mathcal{A}}) \oplus P(\tilde{\mathcal{A}}) \not\geq_e \bar{\emptyset}'$. But $\text{Graph}(g : F(\hat{\mathcal{A}}) \rightarrow F(\tilde{\mathcal{A}})) \geq_e \bar{\emptyset}'$ as $F(\hat{\mathcal{A}}) = \mathcal{B}_1$ and $F(\tilde{\mathcal{A}}) = \mathcal{B}_2$. Thus there can not be an enumeration operator witnessing that F is positive enumerable.

Assume F was enumeration isomorphic to a positive enumerable functor G and that this isomorphism is witnessed by Λ . Then, taking $\hat{\mathcal{A}}, \tilde{\mathcal{A}}$ and $f : \hat{\mathcal{A}} \rightarrow \tilde{\mathcal{A}}$ as in the above paragraph we have that $P(\hat{\mathcal{A}}) \oplus P(\tilde{\mathcal{A}}) \geq_e \text{Graph}(G(f))$. But then

$$P(\hat{\mathcal{A}}) \oplus P(\tilde{\mathcal{A}}) \geq_e \text{Graph}(\Lambda^{P(\hat{\mathcal{A}})} \circ G(f) \circ \Lambda^{P(\tilde{\mathcal{A}})^{-1}}) = \text{Graph}(F(f)) \geq_e \bar{\emptyset}'.$$

This is a contradiction since $\text{deg}_e(P(\hat{\mathcal{A}}) \oplus P(\tilde{\mathcal{A}})) = \mathbf{0}'_e$. □

4 Reductions between arbitrary classes

So far we have seen how we can compare structures with respect to computability theoretic properties. Our notions can be naturally extended to allow the comparison of arbitrary classes of structures.

Definition 24 ([Har+17]). *Let \mathfrak{C} and \mathfrak{D} be classes of structures. The class \mathfrak{C} is uniformly computably transformably reducible, short u.c.t. reducible, to \mathfrak{D} if there are a subclass $\mathfrak{D}' \subseteq \mathfrak{D}$ and computable functors $F : \mathfrak{C} \rightarrow \mathfrak{D}' \subseteq \mathfrak{D}$ and $G : \mathfrak{D}' \rightarrow \mathfrak{C}$ such that F and G are pseudo-inverses.*

Definition 25. *Let \mathfrak{C} and \mathfrak{D} be classes of structures. The class \mathfrak{C} is uniformly (positive) enumerably transformably reducible, short u.e.t., (u.p.e.t.) reducible, to \mathfrak{D} if there is a subclass $\mathfrak{D}' \subseteq \mathfrak{D}$ and (positive) enumerable functors $F : \mathfrak{C} \rightarrow \mathfrak{D}' \subseteq \mathfrak{D}$ and $G : \mathfrak{D}' \rightarrow \mathfrak{C}$ such that F and G are pseudo-inverses.*

Propositions 19 and 20 show that u.p.e.t. and u.c.t. reductions are independent notions.

Corollary 26. *There are classes of structures $\mathfrak{C}_1, \mathfrak{C}_2$ and $\mathfrak{D}_1, \mathfrak{D}_2$ such that*

1. \mathfrak{C}_1 is u.c.t. reducible to \mathfrak{D}_1 but \mathfrak{C}_1 is not u.p.e.t. reducible to \mathfrak{D}_1 .
2. \mathfrak{C}_2 is u.p.e.t. reducible to \mathfrak{D}_2 but \mathfrak{C}_2 is not u.c.t. reducible to \mathfrak{D}_2 .

Similar to Corollary 11 we obtain the equivalence of u.e.t. and u.c.t. reductions.

Corollary 27. *Let \mathfrak{C} and \mathfrak{D} be arbitrary classes of countable structures. Then \mathfrak{C} is u.e.t. reducible to \mathfrak{D} if and only if it is u.c.t. reducible to \mathfrak{D} .*

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