A topological highness notion

Dino Rossegger*

Institute of Discrete Mathematics and Geometry, Technische Universität Wien
dino.rossegger@tuwien.ac.at

Abstract

We isolate a class of continuous functions from Polish spaces to Cantor space, called high functions. We prove characterizations of this class similar to those of high degrees in degree theory, give an application to the theory of topological realizations of countable Borel equivalence relations and answer Problem 4.55 and Problem 4.56 in [Fri+23].

Soare [Soa72] defined the notion of a **high degree** as a Turing degree $\mathbf{d}$ such that $\mathbf{d}' \geq \mathbf{0}''$. In the context of the Δ^0_2 degrees, highness is a notion of strength—for no Δ^0_2 degree, $\mathbf{d}' > \mathbf{0}''$ and so with respect to the Turing jump, high Δ^0_2 degrees are maximally powerful. Since their first mentioning, characterizations in terms of domination properties [Mar66] and enumerations of recursive sets [Joc72]. Since then, highness has been considered in other computability theoretic contexts such as algorithmic randomness [FSY11] and computable structure theory [CFT23; CFT].

We suggest a topological notion of highness for continuous functions $f : X \to 2^\omega$ for X Polish space by proving characterization reminiscent to those by Martin [Mar66], Soare [Soa72] and Jockusch [Joc72] in the degree theory case. In order to state our characterization we need a few preliminaries.

Let (Φ_e) be a listing of the 0-ary Turing operators which correspond to the effectively continuous functions $2^\omega \to \omega$ where ω is thought of as having the discrete topology. Then, given $x \in 2^\omega$ we let **the Turing jump of** x be the characteristic function of the set

$$x' = \{\langle e, n\rangle : \Phi^x_e(n)\downarrow\}.$$

The map $x \mapsto x'$ is Baire class 1, and we will denote it by $\mathsf{J} : 2^\omega \to 2^\omega$. The jump can be iterated, and we denote $\mathsf{J}^n(x)$ as $x^{(n)}$, the nth jump of x, where it is customary to denote the double jump $x^{(2)}$ as x''. Another notion arising in our characterization is the following domination property. Given a bijection $f : X \to 2^\omega$ we say that a function $g : X \times \omega \to \omega$ **dominates computable functions with respect to** f if for every Turing operator e, and all $x \in X$ and all but finitely many $n \in \omega$, $\Phi^{f(x)}_e(n)\downarrow$ implies $\Phi^{f(x)}_{e,g(x,n)}(n)\downarrow$.

Theorem 1. *Suppose that X is a Polish space and $f : X \to 2^\omega$ is continuous. Then the following are equivalent.*

1. *The function $\mathsf{J}^2 \circ f$ is Σ^0_2-measurable.*
2. *For every $P \in \Pi^0_2(2^\omega)$, $f^{-1}(P) \in F_\sigma$.*

If furthermore, X is 0-dimensional, then the following are equivalent to Items 1 and 2:

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3. *There is a continuous function* $g : X \times \omega \to \omega$ *dominating computable functions with respect to* f.
4. *There is a continuous function* $e : X \to 2^{\omega^\omega}$ *so that for every* $x \in X$, $\{e(x)(n) : n \in \omega\} = \{y \leq_T f(x)\}$.

Definition 1. A continuous function $f : X \to 2^\omega$ is **high** if it satisfies Items 1 and 2 in Theorem 1.

The following example is probably the canonical example of a high function.

Example 2. *The inverse of the Turing Jump,* J^{-1} *is a high function.*

Proof. Wlog assume that $(x')^{[1]} = x$, (i.e., the Turing functional with the first index halts exactly if $x(n) = 1$), then J^{-1} is clearly continuous. $J^2 \circ J^{-1} = J$ and thus Σ^0_2-measurable. At last, note that

$$range(J) = \{x \in 2^\omega : \forall n \forall m \Phi^n(m) \downarrow \leftrightarrow x^{[n]}(m) = 1\}$$

where $x^{[n]} = \{\langle n, m\rangle : m \in \omega\}$ is G_δ and thus Polish. □

We will prove Theorem 1 in Appendix A. Our theorem will hopefully peak the interest of computability theorists as it raises other questions such as finding a topological analogue for the degree theoretic notion of lowness, or considering reducibilities between functions where highness corresponds to a notion of strength similar to Turing reducibility.

Our main motivation for this notion was an application to the theory of countable Borel equivalence relations, in particular the theory of topological realizations of Turing equivalence. Recall that an equvialence relation E on a Polish space X is a **topological realization** of an equivalence relation F on a standard Borel space if E and F are Borel isomorphic. The topological realization E is an F_σ **realization** if E is F_σ as a subset of $X \times X$, and it is an **continuous action realization** if it is the orbit equivalence relation of a Polish group acting continuously on X. Clearly, every continuous action realization of a CBER is an F_σ realization. We refer the reader to [Fri+23] for a comprehensive treatment of topological realizations. We note that all these definitions can be made for locally countable Borel quasi-orderings, i.e., quasi-orderings $\preceq$, where $\{y : y \preceq x\}$ is countable for all $x \in X$. See [Wil14] for more on these orderings.

Theorem 3. *Suppose that* $\preceq$ *is a topological realization of* $\leq_T$ *via a high function* $f : X \to 2^\omega$. *Then* $\preceq$ *is an* F_σ *realization.*

Proof. To see this let

$$A_e = \{x \in 2^\omega : \forall n\, \Phi^x_e(n)\downarrow\} \quad \text{and} \quad R_e = \{(x, y) \in 2^\omega \times 2^\omega : \forall n\, (\Phi^x_e(n)\downarrow= y(n))\}.$$

The former set is in $\Pi^0_2(2^\omega)$, and the latter is in $\Pi^0_1(2^\omega \times 2^\omega)$. As $\preceq$ is isomorphic to $\leq_T$ via f we have that

$$x \preceq y \iff \exists e \left(f(x) \in A_e \ \wedge\ (f(x), f(y)) \in R_e\right)$$

As f is high, $f^{-1}(A_e)$ is F_σ and $(f \times f)^{-1}(R_e)$ is closed (as $f \times f$ is continuous). It follows that $\preceq$ is F_σ. □

Corollary 4. *Suppose that E is a topological realization of $\equiv_T$ via a high function $f : X \to 2^\omega$. Then E is an F_σ realization.*

Proof. Let A_e and R_e as above. Then

$$x \mathrel{E} y \iff \exists e_0 \exists e_1 f(x) \in A_{e_0} \wedge f(y) \in A_{e_1} \wedge (f(x), f(y)) \in R_{e_0} \wedge (f(y), f(x)) \in R_{e_1}.$$

By the same argument as above, E is F_σ. □

Theorem 5. *Suppose that $\preceq$ is a continuous action realization of $\leq_T$ on a 0-dimensional space X via continuous $f : X \to 2^\omega$. Then f is high.*

Proof. Suppose that g_i generates $\preceq$ and let $P \in 2^\omega$ be Π^0_2. Then for given $x \in X$, $\{f(g_i(x))\} = \{y \leq_T f(x)\}\}$. As the map $x \mapsto (i, f(g_i(x))$ is continuous, f satisfies Item 4 in Theorem 1 and is thus high. □

Frisch, Kechris, Shinko, and Vidnyánszky asked whether there is a continuous action realization E of Turing equivalence on ω^ω induced by a Baire class 1 bijection $f : 2^\omega \to \omega^\omega$ and whether we can take f such that $f(x) \leq_T x'$ on a cone [Fri+23, Problem 4.56, Problem 4.57]. We answer these questions in the positive.

Theorem 6. *There is a continuous action realization E of $\equiv_T$ on ω^ω induced by a Baire class 1 isomorphism $f : 2^\omega \to \omega^\omega$. In particular $f(x) \leq_T x'$ on a cone.*

Proof sketch. Post's jump theorem implies that $x \equiv_T y$ if and only if $x' \equiv_1 y'$ (the recursive isomorphism equivalence relation). As argued in Example 2, $range(J)$ is a G_δ subset of a 0-dimensional Polish space and thus also 0-dimensional Polish. Furthermore, $\equiv_1$ is the orbit equivalence relation of the group of recursive permutations of S_∞ via the continuous action defined by $(g \cdot x)(n) = x(g(n))$. By [Fri+23, Lemma 4.54], there is $X \subseteq range(J)$ homeomorphic to ω^ω via say h and an isomorphism φ between $\equiv_1$ and $\equiv_1 \restriction X$ that only moves countably many points. This implies that $h \circ \varphi \circ J$ induces a continuous action realization on ω^ω. As φ moves only countably many points, $\varphi \circ J$ is Baire class 1 and thus $f = h \circ \varphi \circ J$ is as required. As $\varphi \circ J(x) = x'$ for all but countably many x and h is continuous, there is $c \in 2^\omega$ such that $x' \geq_T f(x)$ for all $x \geq_T c$. □

At first sight the requirement that f is continuous might appear too strong and not in the spirit of a topological realization as given in [Fri+23]. However, note that the complexity of f^{-1} is crucial, and not the complexity of f. Assuming regularity properties about definable Turing invariant functions, such as Martin's conjecture, we in fact conjecture that any F_σ realization of Turing equivalence must be continuous on a cone. For the special case of compact action realizations we can actually prove this. To do this we need the following easy fact.

Lemma 7. *Assume Martin's conjecture. Suppose that $f : \omega^\omega \to \omega^\omega$ is Borel, injective, and Turing invariant. Then f^{-1} is continuous on a cofinal set.*

Proof. Let $p : 2^\omega \to \omega^\omega$ be the function mapping $x \in 2^\omega$ to the string listing x in order and let $graph : \omega^\omega \to 2^\omega$ be so that $graph(x)$ is the characteristic function of the set $\{\langle n, m\rangle : x(n) = m\}$. Then both p and $graph$ are continuous functions. As f is injective, it is not constant on a cone and neither is $g = graph \circ f \circ p$. So by Martin's conjecture there is a cone c such that $g(x) \geq_T x$, and thus f^{-1} is σ-continuous on c. Playing

Martin's usual game, we obtain a pointed perfect tree such that g^{-1} is continuous on $[T]$ (one can also obtain this using e.g. [MSS16, Lemma 3.5] for the indices of the Turing functionals). Then, as $f^{-1} = p \circ g^{-1} \circ graph$, f^{-1} is continuous on $p([T])$. As both p and $graph$ are degree preserving almost everywhere, f^{-1} is continuous on a cofinal set. □

Proposition 8. *Assume Martin's conjecture. If E is a continuous action realization of $\equiv_T$ via $f : \omega^\omega \to 2^\omega$, then f is continuous on a cofinal set.*

Proof. Suppose that E is an continuous action realization of $\equiv_T$ on ω^ω via $f : \omega^\omega \to 2^\omega$ and let (g_i) continuously generate E. Then there is $z \in \omega^\omega$ so that all the (g_i) are uniformly effectively continuous relative to z. For example, we could take z to be the computable join of the sets $S_i = \{(\sigma, \tau) \in \omega^{<\omega} \times \omega^{<\omega} : g_i^{-1}([\tau]) = [\sigma]\}$. Now consider the closed set $C = \{z \oplus x : x \in \omega^\omega\}$ which is homeomorphic to ω^ω via $c : x \mapsto z \oplus x$, let

$$z \oplus x \ E' \ z \oplus y \iff x \ E \ y,$$

and let $g = c \circ f^{-1}$, i.e., $g(x) = f^{-1}(x) \oplus z$. Clearly, g is Turing invariant and thus by Lemma 7 there is a cofinal set so that g^{-1} is continuous. As $g^{-1} = f \circ c^{-1}$, $f = g^{-1} \circ c$. As the latter is a homeomorphism, f is continuous on a cofinal set. □

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A Proof of Theorem 1

The following lemma is merely a standard recursion-theoretic observation.

Lemma 9. *There is a recursive function $c : \omega \to \omega$ so that for every $x \in 2^\omega$, $n \in x''$ if and only if $\Phi^x_{c(n)}(k) = 1$ for at most finitely many k.*

Proof. Recall that x'' is $\Sigma^0_2(x)$ as a subset of ω, uniformly in x. Thus, there is a recursive predicate R such that

$$n \in x'' \iff \exists m \forall k \exists s R(x \upharpoonright s, n, m, k) \tag{1}$$
$$n \notin x'' \iff \forall m \exists k \exists s \neg R(x \upharpoonright s, n, m, k) \tag{2}$$

Fix n and x. We may view $R(x, n, -)$ as defining a table so that if $n \in x''$, then $R(x, n, -)$ contains a first infinite column. We define $\Phi^x_{c(n)}(0) = 0$ if $\exists s R(x \upharpoonright s, n, 0, 0)$ and $\Phi^x_{c(n)}(0) = 1$ if $\exists s \neg R(x \upharpoonright s, n, 0, 0)$. Assuming that we have defined $\Phi^x_{c(n)}(i)$ we let $j_1 = |\{k \leq i : \Phi^x_{c(n)}(k) = 1\}|$ and $j_0 = \max\{l : \Phi^x_{c(n)}(i-l) \dots \Phi^x_{c(n)}(i) = 0^l\}$. Then

$$\Phi^x_{c(n)}(i+1) = \begin{cases} 0 & \exists s R(x \upharpoonright s, n, j_1, j_0) \\ 1 & \exists s \neg R(x \upharpoonright s, n, j_1, j_0) \end{cases}.$$

If $n \in x''$ there is a least witness m_0 for the outer existential quantifier in Eq. (1). Hence, there is some i_0 so that for all $i > i_0$ $j_1[i] = m_0$ and $j_0[i] > j_0[i-1]$. Thus, by construction, $\Phi^x_{c(n)}$ is only finitely different from the constant 0 string.

On the other hand, if $n \notin x''$, then by definition $j_0[i]$ does not grow monotonically in i, and hence $\lim_i j_1[i] = \infty$, so $\Phi^x_{c(n)}$ has infinitely many 1's. □

Lemma 10. *The following are equivalent.*

1. *$\mathsf{J}^2 \circ f$ is Σ^0_2-measurable.*
2. *For every $P \in \Pi^0_2(2^\omega)$, $f^{-1}(P) \in F_\sigma$.*

Proof. Item 2 $\Longrightarrow$ Item 1. Using the function c given by Lemma 9, consider the sets

$$J^1_n = \{x : n \in x''\} = \{x : \Phi^x_{c(n)} \in \mathrm{Fin}\} \text{ and } J^0_n = \{x : n \notin x''\} = \{x : \Phi^x_{c(n)} \in \mathrm{Inf}\}$$

where Fin is the standard Σ^0_2 set of elements finitely different from the constant 0 string and Inf its complement. Clearly $J^1_n \in \Sigma^0_2$ and $J^0_n \in \Pi^0_2$. Now, suppose that for all Π^0_2 subsets P of 2^ω, $f^{-1}(P) \in F_\sigma$. Then for every n, and the basic open sets $\{x : x(n) = 1\}$ and $\{x : x(n) = 0\}$ we have

$$(\mathsf{J}^2 \circ f)^{-1}(\{x : x(n) = i\}) = f^{-1}(\overbrace{(\mathsf{J}^2)^{-1}(\{x : x(n) = i\})}^{=J^i_n}) \in F_\sigma.$$

As the sets $\{x : x(n) = i\}$ for $i \in \{0,1\}$ form a subbase for 2^ω, $\mathsf{J}^2 \circ f : X \to 2^\omega$ is Σ^0_2-measurable, as required.

Item 1 $\implies$ Item 2. Suppose that $P \subseteq 2^\omega$ is Π^0_2, then there is a recursive predicate R such that

$$x \in P \iff \forall n (\exists m > n) R(x \restriction m, n),$$

which implies that there is an index e_0 for an x-c.e. subset of ω with $n \in W^x_{e_0}$ if and only if $(\exists m > n) R(x \restriction m, n)$ and $x \in P$ if and only if $W^x_{e_0} = \omega$. Now, the set $\mathrm{Tot}^x = \{e : W^x_e = \omega\}$ is a $\Pi^0_2(x)$ index set and by the $\Sigma^0_2(x)$ completeness of x'' as a set of natural numbers there is a computable function $c : \omega \to \omega$ such that

$$e \in \mathrm{Tot}^x \iff x''(c(e)) = 0.$$

Assuming that $\mathsf{J}^2 \circ f$ is Σ^0_2-measurable, $(\mathsf{J}^2 \circ f)^{-1}(\{x : x(c(e_0)) = 0\}) \in F_\sigma$ and chasing the equalities above $x \in f^{-1}(P) \iff \mathsf{J}^2(f(x))(c(e_0)) = 0$, implying that $f^{-1}(P) \in F_\sigma$. □

Lemma 11. *If X is 0-dimensional, then $\mathsf{J}^2 \circ f : X \to 2^\omega$ is Σ^0_2-measurable if and only if there is a continuous function $g : X \times \omega \to \omega$ dominating computable functions.*

Proof. ($\Rightarrow$). We may suppose that $\mathsf{J}^2 \circ f = \lim_s j_s$. As for given $x \in X$, $\mathrm{Tot}^{f(x)} = \{e : \forall n \Phi^{f(x)}_e(n)\downarrow\} \leq_T \mathsf{J}^2(f((x))$ uniformly in $f(x)$ via say Φ_{e_t},

$$e \in \mathrm{Tot}^{f(x)} \iff \lim_s \Phi^{j_s(x)}_{e_t}(e) = 1 \text{ and } e \notin \mathrm{Tot}^{f(x)} \iff \lim_s \Phi^{j_s(x)}_{e_t}(e) = 0$$

Our proof is now a simple adaptation of the proof of Martin's high domination theorem. To define $g(x,s)$ for all $e \leq s$ define

$$t(x,e) = \min\{r > s : \Phi^{j_r(x)}_{e_t}(e) = 0 \vee (\forall n \leq s)\Phi^x_{e,r}(n)\downarrow\}, \quad g(x,s) = \max\{t(x,e) : e \leq s\}.$$

Note that $t(x,e)$ and $g(x,s)$ are always defined and continuous. Also, if Φ^x_e is total, i.e., $\lim_s \Phi^{j_s(x)}_{e_t}(e) = 1$, then $g(x,n) > \Phi^{f(x)}_e(n)$ for almost every n, as required.

($\Leftarrow$). We will produce a family of continuous functions $h_s : X \to 2^\omega$ such that $\lim_s h_s(x) = \mathrm{Tot}^{f(x)}$. Then, as $\mathrm{Tot}^{f(x)} \equiv_T \mathsf{J}^2(f(x))$ uniformly in x via say Φ_e, we have that $\Phi_e \circ h_s : X \to 2^\omega$ is continuous for all s and $\lim_s(\Phi_e \circ h_s) = \mathsf{J}^2 \circ f$. To define $h(x,s)$ we let

$$h(x,s)(n) = \begin{cases} 1 & \text{if } (\forall z < s)\Phi^{f(x)}_{n,g(x,s)}(z)\downarrow \\ 0 & \text{otherwise} \end{cases}$$

The function is clearly continuous as f is continuous. If $n \in \mathrm{Tot}^{f(x)}$, then also $\psi_n(m) = \mu s(\forall z < m)\Phi^{f(x)}_{n,s}(z)\downarrow$ is total. Thus, $\lim_s h(x,s)(n) = 1$. If $\Phi^{f(x)}_n$ is not total, then there is m such that $\Phi^{f(x)}_n(m)\uparrow$ and thus $\lim_s h(x,s)(n) = 0$. It follows that $\lim_s h(x,s) = \mathrm{Tot}^{f(x)}$ as required. □

The following is the relativized version of a Lemma first proved by Jockusch [Joc72].

Lemma 12 ([Joc72])**.** *There is an effectively continuous function $g : 2^\omega \times \omega \to \omega$ such that for every Turing operator Φ_e and $x \in 2^\omega$*

1. Φ_e^x *total implies* $\Phi_{g(x,e)}^x$ *total,*
2. Φ_e^x *not total implies there is no* i *such that* Φ_i^x *is total and* Φ_i^x *extends* $\Phi_{g(x,e)}^x$.

Lemma 13 (Item 4 $\implies$ Item 2)**.** *Suppose that there is a continuous function* $e : X \to 2^{\omega\omega}$ *so that for every* $x \in X$, $\{e(x)(n) : n \in \omega\} = [f(x)]_{\leq_T}$, *then for every* $P \in \Pi_2^0(2^\omega)$, $f^{-1}(P) \in F_\sigma$.

Proof. Let $P \in 2^\omega$ be Π_2^0. Then as explained in the proof of Lemma 10 there is an index e_0 such that $x \in P$ if and only if $e_0 \in Tot^x$. Using Lemma 12 we obtain that

$$x \in f^{-1}(P) \iff e_0 \in Tot^{f(x)} \iff \exists i(e(x)(i) \supseteq \Phi_{g(f(x),e_0)}^{f(x)})$$

and as both f and e are continuous, $f^{-1}(P) \in F_\sigma$. Thus, f is high. □

Lemma 14 (Item 3 $\implies$ Item 4)**.** *Suppose that there is a continuous function* $g : X \times \omega \to \omega$ *dominating computable functions, then there is a continuous function* $e : X \to 2^{\omega\omega}$ *so that for every* $x \in X$, $\{e(x)(n) : n \in \omega\} = \{y \leq_T f(x)\}$.

Proof. Define $e(x)(\langle i, n\rangle)(m) = \Phi_{i,n+g(x)(m)}^{f(x)}(m)$ if $\Phi_{i,n+g(x)(m)}^{f(x)}(k)\downarrow$ for all $k \leq m$ and $e(x)(\langle i, n\rangle)(m) = 0$ otherwise. If $y \leq_T f(x)$, then there is i and n such that for all x $\Phi_{i,n+g(x)(m)}^{f(x)}(m) \downarrow= y(m)$ and thus $y = e(x)(\langle i, n\rangle)$. On the other hand, if $\langle i, n\rangle$ is a column and n sufficiently large, then $\Phi_i = e(x)(\langle i, n\rangle)$, and for all other i, $e(x)(\langle i, n\rangle)$ is finitely nonzero. In any case $y \leq_T f(x)$, and so e is as required. □