Hyperfinite equivalence relations

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In this note we will prove several equivalent conditions for a countable equivalence relation on a standard Borel space to be hyperfinite. These equivalences first appeared in [1].

Theorem 1. *Suppose E is a countable Borel equivalence relation on a standard Borel space X. Then the following are equivalent.*

1. *E is hyperfinite, i.e., $E = \bigcup_{n\in\mathbb{N}} E_n$ with $E_n \subseteq E_{n+1}$ and each E_n finite.*
2. *E is hyperfinite witnessed by $(E_n)_{n\in\mathbb{N}}$ with $|[x]_{E_n}| \leq n$ for all $n \in \mathbb{N}$ and $x \in X$.*
3. *$E \sqsubseteq_B E_0$*
4. *E is generated by a $\mathbb{Z}$ action, i.e., there is a Borel function $f : X \to X$ such that $xEy \iff \exists n\ f^n(x) = y \vee x = y$.*
5. *There is a Borel assignment $[x] \to <_{[x]}$ such that $<_{[x]} \hookrightarrow \mathbb{Z}$. Furthermore, if $[x]$ is infinite, then $<_{[x]} \cong \mathbb{Z}$.*

The equivalence between (3) and (1) is due to Slaman and Steel [2], the equivalence between (1) and (2) is due to Weiss [3] and the others are due to Dougherty, Kechris and Louveau [1].

Before we proceed with the proof of the above theorem recall the Feldman-Moore theorem.

Theorem 2 (Feldman-Moore)**.** *Let E be a countable equivalence relation on a standard Borel space X, then E is induced by the Borel action of a countable group on X, i.e., $E = E_X^G$.*

Fix a Borel ordering $<_X$ on X and say that E is a finite equivalence relation. Then the Feldman-Moore theorem allows us to find largest, and smallest elements in the equivalence classes and the sizes of the equivalence classes in a Borel way. It also implies that finite equivalence relations are smooth.

Theorem 3 (Luzin-Novikov)**.** *Let X, Y be standard Borel spaces and $P \subseteq X \times Y$ be Borel. Suppose that every section $P_x = \{y \in Y : (x, y) \in P\}$ is countable. Then $proj_X(P) = \{x \in X : (\exists y \in Y)(x, y) \in P\}$ is Borel, and P has a* Borel uniformization*. I.e., there is a Borel function $f : Proj_X(P) \to Y$ such that $(x, f(x)) \in P$ for any $x \in proj_X(P)$.*

Lemma 4. *Let E, F be countable Borel equivalence relations on standard Borel spaces X and Y respectively.*

1. *If $X = Y$, F is hyperfinite and $E \subseteq F$ then E is hyperfinite.*
2. *If E is hyperfinite, and $A \subseteq X$ is Borel, then $E \restriction A$ is hyperfinite.*
3. *If $A \subseteq X$ is Borel, $[A]_E = X$ and $E \restriction A$ is hyperfinite, then E is hyperfinite.*
4. *If $E \leq_B F$ and F is hyperfinite, then E is hyperfinite.*
5. *If both E and F are hyperfinite, then $E \times F$ is hyperfinite.*

Proof. (1),(2), and (5) are obvious. For (3), let the hyperfiniteness of E on A be witnessed by (F_n), take $G = \{g_m : m \in \mathbb{N}\}$ such that $E = E_X^G$, and let $m(x)$ be least m such that $g_m \cdot x \in A$ for $x \in X$. Define

$$xE_ny \iff (m(x), m(y) < n \wedge g_{m(x)} \cdot xF_n g_{m(y)} \cdot y) \vee x = y.$$

The E_n are finite equivalence relations, $E_n \subseteq E_{n+1}$ and $E = \bigcup_n E_n$. Thus E is hyperfinite.

For (4), let $f : E \leq_B F$. Then $f(x) = f(y)$, implies xEy and as f is countable to one, $\{((z, x) : f(x) = z\}$ satisfies the conditions for the Luzin-Novikov theorem. So, there is an injective Borel function $g : f(X) \to X$ such that $f \circ g = id$. In particular $g(f(X))$ is Borel, and $[g(f(X))]_E = X$. As $E \restriction g(f(X))$ is Borel isomorphic to $F \restriction f(X)$ we get by (2) that $F \restriction f(X)$ is hyperfinite. Hence, also $E \restriction g(f(X))$ is hyperfinite and thus by (3) E. ☐

This lemma shows (3) implies (1) and in particular implies that any smooth equivalence relation is hyperfinite (Say f witnesses smoothness, then modify f to map to the set of initial segments of $f(x)$ for every x).

Proof of (4) implies (5). If $[x]$ is infinite, define $<$ on $[x]$ by letting $x < y$ iff $\exists n f^n(x) = y$. Otherwise define $<$ using some fixed Borel ordering on X. ☐

Proof of (5) implies (4). Define f by

$$f(x) = \begin{cases} succ(x) & \text{if } succ(x) \downarrow \\ (\mu y \in [x]) pre(y) \uparrow & \text{otherwise} \end{cases}.$$

☐

Proof of (1) implies (5). Given xEy with $x \neq y$ let $n_{xy} = \max m[\neg x E_m y]$ and for every n define $x <_n y$ if the $<_X$-least element of $[x]_{E_n}$ is $<_X$ below the $<_{E_n}$-least element of $[y]_{E_n}$. Let

$$x < y \iff x \neq y \wedge xEy \wedge [x]_{E_{n_{xy}}} \leq_{n_{xy}} [y]_{E_{n_{xy}}}.$$

We just need to observe that $<$ is discrete: Consider an interval $[x, y]$. If zEx and $z \notin ([x]_{E_{n_{xy}}} \cup [y]_{E_{n_{xy}}})$, then $z <_{n_{xz}} x$ or $z >_{n_{xz}} x$. The first case immediately implies that $z < x$ and in the other case if $n_{yz} = n_{xy}$ then z might be in $[x, y]$ but there are only finitely many such candidates or $n_{yz} > n_{xy}$ in which case $z > y$. Thus $[x, y]$ is finite. We can easily modify any order of order type ω or ω^* to have order type $\mathbb{Z}$. ☐

In order to proof that (5) implies (1) we need the following lemma due to Slaman and Steel which is interesting in its own right.

Lemma 5 (Marker lemma)**.** *Every countable Borel equivalence relation E without finite classes admits a* vanishing sequence of markers*, i.e., a sequence of Borel sets $S_0 \supseteq S_1 \ldots$ such that $[S_n]_E = E$ for all n and $\bigcap S_n = \emptyset$.*

Proof. Assume without loss of generality that $X = 2^\omega$ and let $s_n(x) = \mu s \in 2^n[|[x]_E \cap [\![s]\!]| = \infty]$ and let $x \in A_n \iff x \restriction n = s_n(x)$. The sequence $(A_n)_{n \in \mathbb{N}}$ is clearly decreasing and meets every equivalence class of E. Furthermore $\bigcap A_n \cap [x]_E \leq 1$ for all x. Let $S_n = A_n \setminus \bigcap A_n$. ☐

Proof of (5) implies (1). Given a countable Borel equivalence relation E, note that the subequivalence relation $E_{fin} = \{(x, y) : xEy \wedge (x = y \vee [x] \text{ is finite}\}$ is Borel. Hence, we may assume that E does not contain finite equivalence relations. Let $(S_n)_{n \in \mathbb{N}}$ be a vanishing sequence of markers for E and let X_{smooth} be the set of all x such that for some n, $[x] \cap S_n$ has either a least or a greatest element. Then X_{smooth} is a Borel subset of X and $E_{smooth} = E \restriction X_{smooth} \cup Id$ is a smooth subequivalence of E witnessed by the function mapping x to the least or greatest element in $[x] \cap S_n$ such that a least or greatest element exists. For E on $X \setminus X_{smooth}$ define

$$xE_n y \iff x = y \vee (xEy \wedge [x, y] \cap S_n = \emptyset).$$

Clearly, $E_n \subseteq E_{n+1}$ and $E = \bigcup E_n$ and each E_n is finite. As a finite union of hyperfinite equivalence relations is hyperfinite, we have that E is hyperfinite. ☐

At last let us proof that (2) implies (3). This proof is essentially due to Hjorth, see also [4].

Proof of (1) implies (3). Say $(F_n)_{n \in \omega}$ is a hyperfinite witness for E with F_0 the identity. We will reduce E to $E_0(\omega)$ the eventual equality relation on ω^ω. Given n, let $[x]_{F_n} = \{y_0^n, \ldots, y_{k_n}^n\}$ where $y_0^n < \cdots < y_{k_n}^n$ in a fixed Borel ordering of X and fix a bijection $\pi : \omega^{<\omega} \to \omega$. We construct a map $f : X \to \omega^\omega$ as follows.

1. $f(x)(0) = (x \restriction 0 = y_0^0 \restriction 0, \pi(0))$
2. for $n \geq 1$,
$$f(x)(n) = (y_0^n \restriction n, y_1^n \restriction n, \dots y_{k_n}^n \restriction n, \pi(i(n,0), i(n,1), \dots, i(n, k_{n-1})))$$
where $i(n,j)$ is chosen such that $y_j^{n-1} = y_{i(n,j)}^n$ for all $j \leq k_{n-1}$.

We have that if xEy, then $xF_n y$ for all large enough n and hence $f(x)E_0 f(y)$.

To see the backwards direction assume that for given $x, y \in X$ and $n \in \mathbb{N}$, $\forall m \geq n$ $f(x)(m) = f(y)(m)$. For all $m \geq n$, let
$$f(x)(m) = (s_0^m, \dots s_{k_m}^m, \pi(i(m,0), \dots, i(m, k_{m-1}))).$$
For any $j < k_n$ define $J : \mathbb{N}_{\geq n} \to \mathbb{N}$ by
$$J(m) = \begin{cases} j & \text{m=n} \\ i(m, J(m-1)) & m \geq n+1 \end{cases}.$$

By the definitions of f and J, we have that $s_{J(m)}^m \subseteq s_{J(m+1)}^{m+1}$. Thus, $y_j^n = \bigcup_{m \geq n} s_{J(m)}^m$ and hence, since $F_0 = id$ we have that $xF_n y$ and so xEy.

Notice, that since $F_0 = id$, $E \sqsubseteq_B E_0$. □

Proof of (1) implies (2). Let (F_n) witness the hyperfiniteness of E and assume wlog that $F_0 = id$. For each n let
$$X_n = \{x \in X : |[x]_{F_n}| \leq n\}$$
$$X_i = \{x \notin \bigcup_{i<j\leq n} X_j : |[x]_{F_i}| \leq n\}$$

Note that since $X_0 = id$, $X_0 = X - \bigcup_{0<j\leq n} X_j$ and that all X_i are F_j invariant for $j \leq i$. Let $E_n = \bigcup_{i \leq n} F_i \restriction X_i$, then every E_n orbit has at most n elements. To see that $E_n \subseteq E_{n+1}$ let $Y_{n+1}, \dots, Y_0$, be the sequence of sets defined in the definition of E_{n+1}. Say $xE_n y$, then there is $i \leq n$ with $xF_i y$. Since $X_i \subseteq Y_{n+1} \cup \dots Y_i$ there is $j \geq i$ with $x, y \in Y_j$ and hence $xE_{n+1}y$. To see that $E = \bigcup E_n$, just note that if $xF_i y$, since very orbit in F_i is finite, it will get added eventually to some E_n.

□

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