THE DEGREES OF CATEGORICITY ABOVE $\mathbf{0}''$

BARBARA F. CSIMA AND DINO ROSSEGGER

Abstract. We give a characterization of the degrees of categoricity of computable structures greater or equal to $\mathbf{0}''$. They are precisely the *treeable* degrees – the least degrees of paths through computable trees – that compute $\mathbf{0}''$. As a corollary, we obtain several new examples of degrees of categoricity. Among them we show that every degree $\mathbf{d}$ with $\mathbf{0}^{(\alpha)} \leq \mathbf{d} \leq \mathbf{0}^{(\alpha+1)}$ for α a computable ordinal greater than 2 is the strong degree of categoricity of a rigid structure. Another corollary of our characterization partially answers a question of Fokina, Kalimullin, and Miller: Every degree of categoricity above $\mathbf{0}''$ is strong. Using quite different techniques we show that every degree $\mathbf{d}$ with $\mathbf{0}' \leq \mathbf{d} \leq \mathbf{0}''$ is the strong degree of categoricity of a structure. Together with the above example this answers a question of Csima and Ng. To complete the picture we show that there is a degree $\mathbf{d}$ with $\mathbf{0}' \leq \mathbf{d} \leq \mathbf{0}''$ that is not the degree of categoricity of a rigid structure.

Two isomorphic copies of a mathematical structure share the same structural properties and, thus, one usually considers structures up to isomorphism. However, Fröhlich and Shepherdson [FS56] and, independently, Malt'sev [Mal62] showed that isomorphic copies of a structure can behave quite differently with respect to their algorithmic properties. They produced two isomorphic computable fields $\mathcal{K}_1$ and $\mathcal{K}_2$ such that $\mathcal{K}_1$ has a computable transcendence basis, while $\mathcal{K}_2$ fails to have a computable transcendence basis. This leads to the conclusion that there can not be a computable function that is an isomorphism between these two fields [FS56, Corollary 5.51].

Two computable isomorphic structures $\mathcal{A}$ and $\mathcal{B}$ that are computably isomorphic have the same computability theoretic properties and if $\mathcal{A}$ has computable isomorphisms between all its computable isomorphic copies then $\mathcal{A}$ is said to be *computably categorical*. As we have seen even natural structures fail to be computably categorical and often one would like to know how far apart the computable copies of these structures can be with respect to their computability theoretic properties. This is best captured by measuring the Turing complexity of the isomorphisms between computable copies. Towards this, Fokina, Kalimullin, and Miller introduced the following notion [FKM10].

Definition 1. Let τ be a computable vocabulary, $\mathcal{A}$ be a computable τ-structure, and let $(\mathcal{B}_e)_{e\in\omega}$ be an enumeration of all computable τ-structures. The *categoricity*

Date: September 9, 2022.
2020 *Mathematics Subject Classification.* 03C75, 03D45,03E15.
Csima is partially supported by an NSERC Discovery Grant. The work of the second author was supported by the European Union's Horizon 2020 Research and Innovation Programme under the Marie Skłodowska-Curie grant agreement No. 101026834 — ACOSE.

spectrum of $\mathcal{A}$ is the set

$$CatSpec(\mathcal{A}) = \bigcap_{e\in\omega:\mathcal{B}_e\cong\mathcal{A}} \{deg(X) : (\exists f : \mathcal{A} \cong \mathcal{B}_e) X \geq_T f\}.$$

If $\mathbf{d} \in CatSpec(\mathcal{A})$ is a least element, then $\mathbf{d}$ is called the *degree of categoricity of* $\mathcal{A}$.

The study of this notion has been one of the most active areas in computable structure theory in the last decade. See [Fra17] for a survey of developments until 2017. One of the main goals in the area is to obtain a characterization of the Turing degrees that are degrees of categoricity.

Fokina, Kalimullin, and Miller [FKM10] showed that every degree $\mathbf{d}$ d-c.e. in and above $\mathbf{0}^{(n)}$ for some $n \in \omega$ is a degree of categoricity and that $\mathbf{0}^{(\omega)}$ is a degree of categoricity. Csima, Franklin, and Shore [CFS+13] generalized these results to the hyperarithmetic hierarchy, showing that both $\mathbf{0}^{(\alpha)}$ for α a computable limit ordinal and every degree $\mathbf{d}$ d-c.e. in and above $\mathbf{0}^{(\alpha)}$ for α computable successor ordinal are degrees of categoricity. The former result was later improved by Csima, Deveau, Harrison-Trainor, and Mahmoud [Csi+18] to degrees c.e. in and above limit ordinals.

It is known that not every Turing degree is a degree of categoricity. Csima, Franklin, and Shore [CFS+13] showed that every degree of categoricity must be hyperarithmetic and Anderson and Csima [AC12] provided several examples of degrees that are not degrees of categoricity. For example, they obtained a Σ^0_2 degree that is not a degree of categoricity.

Recently, Csima and Ng [CN22] showed that every Δ^0_2 degree is a degree of categoricity. They asked whether for every computable ordinal α, every Turing degree $\mathbf{d}$, $\mathbf{0}^{(\alpha)} \leq \mathbf{d} \leq \mathbf{0}^{(\alpha+1)}$ is a degree of categoricity.

The main goal of this article is a characterization of the degrees of categoricity on the cone above $\mathbf{0}''$. We characterize these degrees by showing that they are exactly the degrees of Turing-least paths through computable trees in ω^ω (Corollary 4). Using classical results about Π^0_1 function singletons we then obtain that every degree $\mathbf{d}$ with $\mathbf{0}^{(\alpha)} \leq \mathbf{d} \leq \mathbf{0}^{(\alpha+1)}$ for α a computable ordinal greater or equal to 2 is the degree of categoricity of a rigid structure. Building on a construction by Csima and Ng [CN22] that showed that every Δ^0_2 degree is a degree of categoricity we complete the picture by showing that every degree $\mathbf{d}$, $\mathbf{0}' \leq \mathbf{d} \leq \mathbf{0}''$ is a degree of categoricity. We thus obtain a positive answer to their first question (Theorem 13). We also obtain more exotic examples. Using Harrington's example of a non-arithmetic Π^0_2 singleton [Har76], we obtain degrees of categoricity that are not between $\mathbf{0}^{(\alpha)}$ and $\mathbf{0}^{(\alpha+1)}$ for any computable ordinal (Corollary 9).

Another driving question in the area is whether for every degree of categoricity $\mathbf{d}$ there are computable structures $\mathcal{A}$ and $\mathcal{B}$ such that $\mathbf{d}$ is the least degree computing isomorphisms between $\mathcal{A}$ and $\mathcal{B}$. Such degrees are called *strong* degrees of categoricity and at the time of writing all known examples of degrees of categoricity are strong. One corollary of our characterization partially answers this question: Every degree of categoricity above $\mathbf{0}''$ is strong (Corollary 5).

All of the above results are corollaries of our main theorem whose proof is a modification of a recent construction of Turetsky that coded paths through trees into the automorphisms of a structure [Tur20]. In order to state it, we need to recall a bit of notation. Given a tree $T \subseteq \omega^{<\omega}$ we denote by $[T] \subseteq \omega^\omega$ the set of paths through T. For two sets $P, Q \subseteq \omega^\omega$ we say that P is Muchnik reducible to Q, $P \leq_w Q$, if for every $q \in Q$ there is $p \in P$ with $p \leq_T q$. We can computably translate elements of ω^ω to elements of 2^ω and sets of natural numbers and thus will use Muchnik reducibility to compare sets of these types. The *computable dimension* of a structure is the number of computable copies that are not computably isomorphic. This number is either finite or ω and constructions of examples of finite computable dimension usually involve heavy computability theoretic machinery. The first example of such a structure was given by Goncharov [Gon80]. The following is our main theorem from which most other results in this article are derived.

Theorem 1. *Let $T \subseteq \omega^{<\omega}$ be a computable tree such that $[T] \geq_w \{\emptyset''\}$. Then there is a computable structure $\mathcal{S}_1$ with computable dimension 2 such that*

$$CatSp(\mathcal{S}_1) = \{\deg(X) : \{X\} \geq_w [T]\}.$$

Furthermore, if $[T]$ is a singleton, then $\mathcal{S}_1$ is rigid.

Theorem 1 is proven in Section 1.1 and its corollaries are derived in Section 1.2. In Section 2 we modify the construction of Csima and Ng to show that every degree $\mathbf{d}$ with $\mathbf{0}' \leq \mathbf{d} \leq \mathbf{0}''$ is the degree of categoricity of a structure.

At last, in Section 3 we generalize a construction by Bazhenov and Yamaleev [BY17] to show that there is a Turing degree $\mathbf{d}$, $\mathbf{0}' \leq \mathbf{d} \leq \mathbf{0}''$ that is not the degree of categoricity of a rigid structure. This shows that the lower bound in Theorem 1 can not be improved.

1. Characterizing degrees of categoricity above $\mathbf{0}''$

1.1. **Degrees of categoricity and paths.** Recall that for two sets $X, Y \subseteq \omega^\omega$, X is *Muchnik reducible* to Y, $X \leq_w Y$ if for every $y \in Y$ there is $x \in X$ such that $x \leq_T y$. Turetsky proved that given a computable tree $T \subseteq \omega^{<\omega}$ there is a computable, computably categorical structure $\mathcal{S}$ such that the paths of T and the non-trivial automorphisms of $\mathcal{S}$ are Muchnik equivalent modulo $\mathbf{0}''$ [Tur20]. In other words,

$$\{\emptyset'' \oplus f : f \in [T]\} \equiv_w \{\emptyset'' \oplus \nu : \nu \in Aut(\mathcal{S}) \setminus id\}.$$

Turetsky also exhibited how the structure $\mathcal{S}$ can be adapted to obtain a structure $\mathcal{S}_1$ that is not hyperarithmetically categorical and has computable dimension 2. For this structure, it is the case that the isomorphisms between the two copies witnessing the computable dimension are Muchnik equivalent to the paths through T modulo $\mathbf{0}''$.

If one can eliminate the $\mathbf{0}''$ in these results one obtains a coding technique that allows the coding of paths through trees into categoricity spectra of structures of computable dimension 2. If T has a unique path f, i.e., f is a Π^0_1 function singleton, then an analysis of Turetsky's construction shows that the structure $\mathcal{S}_1$ obtained from T is rigid.

Our first result improves on Turetsky's by eliminating $\emptyset''$ on the right. We obtain this result by adding requirements to Turetsky's construction with the aim of coding $\emptyset''$ into the presentation of $\mathcal{S}$.

Lemma 2. *Let $T \subseteq \omega^{<\omega}$ be a computable tree. Then there is a computable, computably categorical structure $\mathcal{S}$ such that*

$$\{\nu : \nu \in (Aut(\mathcal{S}) - \{id\})\} \equiv_w \{f \oplus \emptyset'' : f \in [T]\}.$$

In particular, $|[T]| = 1$ *if and only if* $|Aut(\mathcal{S}) - \{id\}| = 1$*.*

Lemma 2 is the main ingredient to our proof of Theorem 1. Its proof is a modification of Turetsky's infinite injury construction [Tur20, Theorem 2]. We give a full description of the structure we are going to construct and the construction, but only sketch the verification in the sense that we prove that our construction works and gives the desired results if Turetsky's initial construction works. Thus, the reader is encouraged to read this together with Turetsky's proof for a full verification.

Proof sketch. Given a tree T the vocabulary of $\mathcal{S}$ consists of a unary relations U, $(W_\sigma)_{\sigma\in\omega^{<\omega}}$, and $(S_n)_{n\in\omega}$ (V_n in Turetsky's construction), binary relations P and $(E_n)_{n\in\omega}$, and a unary function f. We denote by $[\omega]^{<\omega}$ the set of all finite subsets of the natural numbers. The universe of $\mathcal{S}$ will be $[\omega]^{<\omega} \times \omega^{<\omega} \sqcup C$ where C is an infinite computable set. The relation symbols U and f and the set C are used to help with the following issue. During the construction we will want $S_n(x)$ to hold for larger and larger n on elements of $[\omega]^{<\omega} \times \omega$. The issue is, that this would result in a c.e. structure and not a computable structure. To overcome this we do not define the S_n directly on the x but rather using an element $y \in C$ that is associated to x by f, i.e., $f(x) = y$. Formally, we will have the following.

The relation U is used to identify C, i.e. $U^{\mathcal{S}} = C$. We will have that ${S_n}^{\mathcal{S}} \cap \overline{C} = \emptyset$ while none of the relations W_σ, E_n and P hold on any element in C. We will also delcare $f^{\mathcal{S}}(x) = x$ for all $x \notin C$ and for each $x \in C$ $f(x) \notin C$. Furthermore, for each $x \in C$ there will be a unique n such that $S_n(x)$ holds. During the construction we will ignore C, U and f and simply declare that $S_n(x)$ holds for some element x in $[\omega]^{<\omega} \times \omega^{<\omega}$. What this means is that we pick an unused element $y \in C$, set $f(y) = x$ and $S_n(y)$ if no element with these properties exists at that point of the construction.

With the exception of the relation S_n, our structure looks exactly as the one constructed in [Tur20, Theorem 2]. For $(F,\tau),(G,\rho) \in [\omega]^{<\omega} \times \omega^{<\omega}$ we have that:

- $W_\sigma^{\mathcal{S}}((F,\tau))$ if and only if $\sigma = \tau$
- ${E_i}^{\mathcal{S}}((F,\tau),(G,\rho))$ if and only if $\tau = \rho$ and $F\Delta G = \{i\}$
- $P^{\mathcal{S}}((F,\rho),(G,\tau))$ if and only if $\rho^\frown i$ for some i and one of the following holds:
  - $i \notin F$ and $|G|$ is even or
  - $i \in F$ and $|G|$ is odd.

It is convenient to think of each W_σ slice of $\mathcal{S}$ as an infinite dimensional hypergraph with the edge relations given by the E_i. One can then prove that the automorphisms of these slices in the reduct $(E_i)_{i\in\omega}$ are exactly the maps of the form $(F,\sigma) \mapsto (F\Delta H,\sigma)$ for some fixed $H \in [\omega]^{<\omega}$ [Tur20, Claim 2.1]. So, let g be a non-trivial automorphism of the structure we have defined so far, then g acts non-trivially on one of the $W_\sigma^{\mathcal{S}}$, i.e., $g : (F,\sigma) \mapsto (F\Delta H,\sigma)$ for $H \neq \emptyset$. In particular $g : (\emptyset,\sigma) \mapsto (H,\sigma)$. The P predicate forces g to act on $\sigma^\frown i$ for $i \in H$. This can be used to prove that the automorphisms of this presentation of $\mathcal{S}$ are Muchnik equivalent to the paths through T.

The only remaining issue is that $\mathcal{S}$ has many computable presentations and so far we have only controlled the automorphisms of one such presentation. This is where the S_n predicates come in. They are used to ensure that $\mathcal{S}$ is computably categorical. This process is what requires a $\mathbf{0}''$ priority argument and thus introduces the

$\emptyset''$ in the original construction. The construction will give rise to a $\mathbf{0}''$ computable tree $Q \subseteq \omega^{<\omega}$ that is $\mathbf{0}''$ isomorphic to T such that for $\sigma \in Q$, $S_n((F,\sigma))$ will hold for all $n \in \omega$ and $F \in [\omega]^{<\omega}$ and for $\sigma \notin Q$, there is an n such that $S_n((F,\sigma))$ holds if and only if $F = \emptyset$. One can show that the automorphisms of $\mathcal{S}$ are then Muchnik equivalent to the paths through Q. In particular, there is a $1-1$ correspondence between $Aut(\mathcal{S}) \setminus id$ and $[Q]$ [Tur20, Claim 2.2].

However, Q is only $\mathbf{0}''$ isomorphic to the original tree T. This is why we modify our construction by forcing that $\sigma \in Q$ with $\sigma(i)\downarrow$ satisfies $2 \mid \sigma(i)$ if and only if $i \in \emptyset''$. Now, given an automorphism of $\mathcal{S}$ we can compute a path $f \in [Q]$ and from $f \oplus \emptyset''$ we can compute a path through T by computing the isomorphism. However, the condition on Q guarantees that $f \geq_T \emptyset''$ and thus $\{\nu : \nu \in (Aut(\mathcal{S}) \setminus \{id\})\} \geq_w \{f \oplus \emptyset'' : f \in [T]\}$. That $\{\nu : \nu \in (Aut(\mathcal{S}) \setminus \{id\})\} \leq_w \{f \oplus \emptyset'' : f \in [T]\}$ follows from a similar argument. Given $f \in [T]$, $f \oplus \emptyset''$ can compute an element of $[Q]$ and thus a non-trivial automorphism of $\mathcal{S}$.

Construction. Turetsky's construction had a requirement G, requirements N_π for every $\pi \in T$ and M_i for every $i \in \omega$. We add requirements R_i for $i \in \omega$ where R_i has the aim to code the membership of i in $\emptyset''$.

We use Turetsky's order of requirements, interleaving our requirements R_i such that $R_i \leq N_\pi$ for all π with $|\pi| = i+1$. The potential outcomes for R_i are $\infty \leq fin$. Using the fact that $\emptyset''$ is Σ^0_2 we fix a computable function f such that $x \in \emptyset''$ if and only if $W_{f(x)}$ is a proper initial segment of ω and $W_{f(x)} = \omega$ otherwise. We will also need to modify Turetsky's strategy for N_π where $\pi \neq \emptyset$. The goal of the strategy M_i is to ensure that if τ is a strategy on the true path (i.e., the left most path of the priority tree visited infinitely often) and $\mathcal{M}_i$ – the ith computable structure in a computable enumeration of the structures in the language of $\mathcal{S}$ – is isomorphic to $\mathcal{S}$, then $\mathcal{M}_i$ is computably isomorphic to $\mathcal{S}$. This strategy is unmodified and we thus omit it here.

At the end of every stage s, declare $S_k((\emptyset,\sigma))$ for every $k < s$ and $\sigma \in s^{<s}$ not chosen by any strategy.

Strategy for R_i. Suppose τ is a strategy for R_i visited at stage s. Check if $W_{f(x),s} \supset W_{f(x),t}$ where t is the last stage that τ acted if it exists. If so let the outcome of τ be ∞. Otherwise finish with outcome fin.

Strategy for $N_{\pi^\frown i}$. Suppose τ is a strategy for the requirement $N_{\pi^\frown i}$. Then there is a unique $\rho \subset \tau$ such that ρ is a strategy for N_π and a unique $\sigma \subset \tau$ such that σ is a strategy for $R_{|\pi|}$. Suppose s_0 is the first stage such that τ is visited and that ρ has declared ζ to be the image of π. If the outcome of σ is ∞, then choose the least even $m > s_0$ not mentioned in the construction and declare $\zeta^\frown m$ to be the image of $\pi^\frown i$. Otherwise choose the least odd $m > s_0$ not mentioned in the construction and declare $\zeta^\frown m$ to be the image of $\pi^\frown i$.

At every stage s when τ is visited, declare $S_n((\emptyset,\zeta^\frown m))$ for the least n such that $S_n((\emptyset,\zeta^\frown m))$ does not hold at this stage and declare $S_k((F,\zeta^\frown m))$ for all $k < n$ and $F \subseteq \{0,\dots,s\}$. Take the outcome `outcome`.

Verification. Let P be the tree of strategies of this construction and let T be the tree of strategy of [Tur20, Theorem 2]. Given $\sigma \in P$, define its reduct $\sigma_r \in T$ by deleting the occurences of R-strategies from σ. I.e., assume without loss of generality that no R_i strategy is succeeded by any R_j strategy for $i,j \in \omega$ and define

$$k(-1) = -1 \text{ and } k(i) = \begin{cases} k(i-1)+1 & \sigma(k(i-1)+1) \notin \{\infty, fin\} \\ k(i-1)+2 & \text{otherwise} \end{cases}$$

for $i \geq 0$ and let $\sigma_r(i) = \sigma(k(i))$. Let $P^- = \bigcap_{i\in\omega} \{\sigma \in P : \sigma(|\sigma|-1) \notin \{\infty, fin\}\}$. While P^- is not a set-theoretic tree, it still is a tree as a partial order. It is then easy to see that the map $h : P^- \to T : \sigma \mapsto \sigma_r$ is a homomorphism of trees that preserves the priority ordering, i.e., for $\sigma, \tau \in P^-$

$$\sigma \preccurlyeq \tau \implies h(\sigma) \preccurlyeq h(\tau) \text{ and } \sigma \leq^P \tau \implies h(\sigma) \leq^T h(\tau).$$

Claim 2.1. *If a path $f \in [T]$ is visited infinitely often, then there is a path $g \in [P^-] = [P]$ such that $h(g) = f$. Moreover, if f is least in $[T]$ with respect to the priority ordering, then g can be taken least in $[P]$.*

Proof. Run a strategy σ for R_i in isolation and let the true outcome of σ be the leftmost outcome of σ that occurs infinitely often. Clearly the true outcome is ∞ if and only if W_i is infinite and fin otherwise. Furthermore, it does not depend on the outcome of any other strategies. Denote this outcome by t_i. Given a strategy $\sigma \in T$ that is visited infinitely often note that for $\tau, \rho \in h^{-1}(\sigma)$, $|\tau| = |\rho|$. Consider the string $\tau \in h^{-1}(\sigma)$ obtained by filling the gaps by t_i, i.e., if $\tau(j)$ is the ith occurence of ∞ or fin in τ, then $\tau(j) = t_i$. Clearly, τ is visited infinitely often. Also, no string that was visited infinitely often in $h^{-1}(\sigma)$ can be to the left of τ as they can only differ on positions containing ∞ or fin. The claim now follows by induction on the length of τ. □

By Claim 2.1 we have that if f is the true path in our construction, then $h(f)$ is the true path in the original construction. We have to prove that the strategies fulfil their goals, i.e., the M_i strategy ensures that if $\mathcal{M}_i \cong \mathcal{S}$, then $\mathcal{M}_i$ is computably isomorphic to $\mathcal{S}$. The verification of this is exactly as in Turetsky's original proof and thus omitted here. We also have to show that the N_π strategies together ensure that we compute isomorphisms through a tree Q isomorphic to T. At last we have to prove that every automorphism computes $\emptyset''$. This is done via the following claims. Let f be the true path through the construction, that is the lexicographically least path in the priority tree that is visited infinitely often during the construction.

Claim 2.2. *Let $(F, \sigma) \in \mathcal{S}$ with $F \neq \emptyset$. Then $S_n^{\mathcal{S}}(F, \sigma)$ for all $n \in \omega$ if and only if*

1. *σ is the image of π as declared by some N_π strategy $\tau \subset f$,*
2. *for $i \in \omega$ such that $\pi(i)\downarrow$, $W_{f(i)} = \{1 \dots n\}$ if $2 \nmid \sigma(i)$ and $W_{f(i)} = \omega$ otherwise.*

Proof. If σ is the image of N_π as declared by the unique N_π strategy $\tau \subset f$, then $S_n^{\mathcal{S}}(F, \sigma)$ for all $F \subseteq [\omega]^{<\omega}$ and $n \in \omega$.

On the other hand, if $S_n^{\mathcal{S}}(F, \sigma)$ for all $n \in \omega$, then there is a unique N_π strategy τ with σ declared the image of π that acts infinitely often. We have that $\tau \subset f$ because if τ was to the right of the true path then it could grow a single σ only finitely many times before being initialized. Notice that by induction the map $\pi \mapsto \sigma$ mapping N_π strategies to their images σ is Lipschitz. Thus $\sigma(i)\downarrow$ for all i such that $\pi(i)$ is defined. Furthermore, there are unique $\rho_i \subseteq \tau$ such that ρ_i is a strategy for R_i. By τ being on the true path ρ_i has outcome fin if and only if $W_{f(i)} = \{1 \dots n\}$ for some $n \in \omega$ and ρ_i has outcome ∞ if and only if $W_{f(i)} = \omega$. By construction $2 \nmid \sigma(i)$ in the first case and $2 \mid \sigma(i)$ in the second case. □

Claim 2.3. *For every non-trivial automorphism g of $\mathcal{S}$, $g \geq_T \emptyset''$.*

Proof. By [Tur20, Claim 2.1] for every automorphism g and every σ there is $H_\sigma \in [\omega]^{<\omega}$ such that $g((F, \sigma)) = (F \Delta H_\sigma, \sigma)$ for all $F \in [\omega]^{<\omega}$. If g is non-trivial then there must be σ such that $g((\emptyset, \sigma)) = (H_\sigma, \sigma)$ for some $H_\sigma \neq \emptyset$ with $S_n^{\mathcal{S}}((H_\sigma, \sigma))$ for all $n \in \omega$. Now, by Claim 2.2, σ codes an initial segment of $\emptyset''$. We can now compute $\emptyset''$ iteratively as follows. Pick the least $i \in H_\sigma$. We have that $P^{\mathcal{S}}((\emptyset, \sigma), (\emptyset, \sigma ^\frown i))$ but $\neg P^{\mathcal{S}}((H_\sigma, \sigma), (\emptyset, \sigma ^\frown i))$, so in particular $g((\emptyset, \sigma ^\frown i)) = (H_{\sigma ^\frown i}, \sigma ^\frown i)$ for some $H_{\sigma ^\frown i} \neq \emptyset$. As $S_n^{\mathcal{S}}(\emptyset, \sigma_i)$ for all n, also $S_n^{\mathcal{S}}(H_{\sigma ^\frown i}, \sigma ^\frown i)$ for all n. Thus, the strategy declaring $\sigma ^\frown i$ has been visited infinitely often and so $\sigma ^\frown i \in Q$. By Claim 2.2, $2|i$ if and only if $i \notin \emptyset''$. Iterate this procedure to compute initial segments of $\emptyset''$ of arbitrary length and thus $\emptyset''$. □

Let σ_τ be the image of τ as declared by some N_τ strategy on f and σ_ρ be the image of ρ as declared by some N_ρ strategy on f. Then by choice of σ_τ, σ_ρ during the construction $\sigma_\tau \preccurlyeq \sigma_\rho$ if and only if $\tau \preccurlyeq \rho$. Let Q be the set of images of the N_π strategies, then Q is a tree isomorphic to T. Notice that $\mathbf{0}''$ can compute the true path and thus $\{f \oplus \emptyset'' : f \in [T]\} \equiv_w [Q]$. To finish the proof it remains to show the following.

Claim 2.4. *The sets $\{\nu : \nu \in (Aut(\mathcal{S}) \setminus \{id\})\}$ and $[Q]$ are in bijection and Muchnik equivalent.*

Proof. This is just [Tur20, Claim 2.2]. □

□

Modifying the structure $\mathcal{S}$ slightly as in [Tur20, Theorem 3] we can obtain a proof of Theorem 1.

Proof of Theorem 1. Let $\mathcal{S}$ be the structure from Lemma 2. The structure $\mathcal{S}_1$ is obtained by adding two new elements a_{odd} and a_{even} to $\mathcal{S}$ and a new constant symbol c to its vocabulary. The structure $\mathcal{S}_1$ satisfies

- $P^{\mathcal{S}_1}(a_{even}, (F, \langle\rangle))$ if and only if $|F|$ is even,
- $P^{\mathcal{S}_1}(a_{odd}, (F, \langle\rangle))$ if and only if $|F|$ is odd,
- $c^{\mathcal{S}_1} = a_{even}$,

and no other relation holds on a_{even} or a_{odd}. Let $\mathcal{S}_2$ be identical to $\mathcal{S}_1$ except that $c^{\mathcal{S}_2} = a_{odd}$. Turetsky proved that this structure has computable dimension 2 and that every path through Q computes an isomorphism between $\mathcal{S}_1$ and $\mathcal{S}_2$ and vice versa. We will include this here for completeness and include a proof that this structure is rigid if $[Q]$ is a singleton. Theorem 1 then follows from the fact that $Q \cong T$ and if $[T] \geq_\omega \{\emptyset''\}$, then $[T] \equiv_w [Q]$.

Note that given $\mathcal{B}$ isomorphic to $\mathcal{S}_1$ via f, the substructure of $\mathcal{B}$ that does not contain $f(a_{even})$, $f(a_{odd})$ in the language of $\mathcal{S}$ is isomorphic to $\mathcal{S}$. As $\mathcal{S}$ is computably categorical, there is a computable isomorphism g between this substructures. This isomorphism can be extended to an isomorphism between $\mathcal{S}_1$ and $\mathcal{B}$ by letting $g(a_{even}) = f(a_{even})$ and $g(a_{odd}) = f(a_{odd})$. If $c^{\mathcal{B}} = a_{even}$, this isomorphism is clearly computable. Otherwise, using a similar argument we obtain a computable isomorphism between $\mathcal{B}$ and $\mathcal{S}_2$.

Notice that every isomorphism g between $\mathcal{S}_1$ and $\mathcal{S}_2$ induces a non-trivial automorphism of $\mathcal{S}$ as $g(a_{even}) = a_{odd}$ and hence $g((\emptyset, \langle\rangle)) = (F, \langle\rangle)$ for some nonempty set F. But if $[T]$ is a singleton, then $\mathcal{S}$ has exactly one non-trivial automorphism. Note, that in this case there is no non-trivial automorphism of $\mathcal{S}_1$, as the only such automorphism of $\mathcal{S}_1$ would map $(\emptyset, \langle\rangle) \mapsto (\{n\}, \langle\rangle)$ and hence $c^{\mathcal{S}_1} = a_{even} \mapsto a_{odd} \neq C^{\mathcal{S}_1}$. Thus, $\mathcal{S}_1$ is rigid. □

1.2. **Treeable degrees.**

Definition 2 ([BKY18])**.** For any computable structure $\mathcal{S}$, the *spectral dimension* of $\mathcal{S}$ is the least $k \leq \omega$ such that there exists an enumeration $(\mathcal{A}_i, \mathcal{B}_i)_{i<k}$ of pairs of computable copies of $\mathcal{S}$ such that

$$CatSpec(\mathcal{S}) = \bigcap_{i<k} \{deg(X) : (\exists f \leq_T X) f : \mathcal{A}_i \cong \mathcal{B}_i\}.$$

A degree $\mathbf{d}$ is a *strong degree of categoricity* if there is a structure $\mathcal{S}$ of spectral dimension 1 with degree of categoricity $\mathbf{d}$.

One of the main open questions in the area is whether every degree of categoricity is strong. Using Theorem 1 we will answer this question for degrees on the cone above $\mathbf{0}''$.

Recall that $P \subseteq \omega^\omega$ is a Π^0_1 class if there is a computable tree T with $P = [T]$.

Definition 3. A Turing degree $\mathbf{d}$ is *treeable* if there is a Π^0_1 class P such that there exists $p \in P$ with $deg(p) = \mathbf{d}$ and $P \geq_w \{p\}$.

Lemma 3. *Every degree of categoricity is treeable.*

Proof. Let $\mathcal{A}$ be a computable structure computed by Φ_e and let $I(\mathcal{A}) = \{j : \Phi_j \cong \mathcal{A}\}$ be the set of indices of its computable copies. Then

$$deg(f) \in CatSpec(\mathcal{A}) \iff \forall j (\exists g \leq_T f)(j \in I(\mathcal{A}) \to g : \Phi_e \cong \Phi_j)$$

If the above characterization was Σ^1_1 we would get a computable tree witnessing the treeability of the categoricity spectrum of $\mathcal{A}$. Unfortunately, the set $I(\mathcal{A})$ is only guaranteed to be Σ^1_1 and thus the characterization is in general Σ^1_2.

However, Csima, Franklin, and Shore [CFS+13] showed that every degree of categoricity is hyperarithmetic. If $\mathcal{A}$ has degree of categoricity $\mathbf{d}$, then $I(\mathcal{A})$ is arithmetical in $\mathbf{d}$ and thus Δ^1_1. So, if $\mathbf{d}$ is the degree of categoricity of a structure $\mathcal{A}$, then $CatSpec(\mathcal{A})$ is Σ^1_1 and thus the set of paths through a computable tree. This tree has $\mathbf{d}$ as its Turing least path and so $\mathbf{d}$ is treeable. □

Clearly, if a structure $\mathcal{S}$ has computable dimension 2, then it has spectral dimension 1 and thus any degree of categoricity of a structure with computable dimension 2 must be strong. We thus obtain the following from Theorem 1.

Corollary 4. *Let* $\mathbf{d} \geq \mathbf{0}''$. *Then* $\mathbf{d}$ *is a degree of categoricity if and only if it is treeable.*

Corollary 5. *Every degree of categoricity above* $\mathbf{0}''$ *is strong.*

Question 1. *Is every treeable degree the degree of categoricity of a structure?*

Which degrees are treeable? Recall that $f \in \omega^\omega$ is a Π^0_1 function singleton if there is a tree T with $[T] = \{f\}$. Clearly degrees that contain Π^0_1 function singletons are treeable degrees and one can find plenty of examples of those in the literature. Let us summarize.

Abusing notation we say that a degree $\mathbf{d}$ is a Π^0_1 function singleton if it contains a Π^0_1 function singleton. We now list examples of classes of degrees that are Π^0_1 function singletons and thus by Theorem 1 degrees of categoricity of rigid structures with computable dimension 2.

Most of these examples are obtained using the following trick due to Jockusch and McLaughlin [JM69]. Let X be a Π^0_2 singleton, i.e., the only solution to a Π^0_2 predicate. Then it is defined by a formula $\forall u \exists v R(u,v,X)$ which can be rewritten as

$$\forall u \exists (v > u) \forall (x < v) R(u, v, \chi_X \upharpoonright x)$$

where $R(u,v,\chi_X \upharpoonright x)$ implies $R(u,v,\chi_X \upharpoonright y)$ for $y < x$. Let $f(u)$ be the string $\sigma \preceq \chi_A$ of length $\mu v[R(u,v,\sigma)]$. Then $f \leq_T A$ by definition and $A \leq_T f$ as the length of the strings in the range of f is unbounded. The function f itself is a Π^0_1 singleton as it is the unique solution to the equation

$$\forall x (f(x) \in 2^{<\omega} \wedge R(x, |f(x)|, f(x))) \wedge \forall (\tau \preceq f(x)) \neg R(x, |\tau|, \tau).$$

So, a degree $\mathbf{d}$ contains a Π^0_1 function singleton if and only it contains a Π^0_2 singleton. The following theorem is well known.

Lemma 6 (folklore, see [Sac90, Section II.4.]). *For all computable ordinals α, $\mathbf{0}^{(\alpha)}$ is a Π^0_1 function singleton.*

Lemma 7 (folklore, see [Odi99]). *If $\mathbf{d}$ is a Π^0_1 function singleton, then so is every $\mathbf{c}$ with $\mathbf{d} \leq \mathbf{c} \leq \mathbf{d}'$.*

Proof. Let $D \in \mathbf{d}$ be a Π^0_2 singleton defined via $\varphi(X)$ and let $C \in \mathbf{c}$, then C is limit computable from D, say via the Δ^0_2 function $\psi(X,y)$. Consider the set $C \oplus D$. It is the unique solution to the formula

$$\varphi(X_{odd}) \wedge (\psi(X_{odd}, y) \leftrightarrow 2y \in X)$$

where $X_{odd} = \{x : 2x+1 \in X\}$. Using the above trick we get that the degree $\mathbf{c}$ of $C \oplus D$ is a Π^0_1 function singleton. □

Combining Lemma 6 for $\alpha \geq 2$ with Lemma 7 and Theorem 1 we get a new class of degrees that are degrees of categoricities and answer a question posed by Csima and Ng [CN22].

Corollary 8. *Fix $\alpha > 2$ and let $\mathbf{d}$ be such that $\mathbf{0}^{(\alpha)} \leq \mathbf{d} \leq \mathbf{0}^{(\alpha+1)}$, then there is a computable rigid structure $\mathcal{S}$ with computable dimension 2 such that $dgCat(\mathcal{S}) = \mathbf{d}$.*

We can get even more. Harrington [Har76] showed the existence of a Π^0_2 singleton A that is not arithmetical and such that $A^{(n)} \not\geq \emptyset^{(\omega)}$ for all n. While the degree of A itself might not satisfy the conditions of Corollary 4 we can take A'' to obtain the following.

Corollary 9. *There exists a non-arithmetic degree $\mathbf{d}$, $\mathbf{d} \not\geq \mathbf{0}^{(\omega)}$, such that for all $n \in \omega$ and $\mathbf{c} \in [\mathbf{d}^{(n)}, \mathbf{d}^{(n+1)}]$ is the degree of categoricity of a rigid structure with computable dimension 2.*

It is also not hard to give examples of degrees that are not treeable degrees by looking at sufficiently generic degrees. This is not surprising as examples of degrees that are not degrees of categoricity are usually obtained by building degrees that are sufficiently generic [AC12; FS14].

Lemma 10. *Let* $\mathbf{d}$ *be sufficiently generic, then* $\mathbf{d}$ *is not treeable.*

Proof. Let T be a tree with Turing-least element of degree $\mathbf{d}$ and let

$$\varphi(x) \iff \exists i \forall n\, \Phi_i^x \restriction n \in T \land \forall i (\forall n\, \Phi_i^x \restriction n \in T \to x \leq_T \Phi_i^x) \tag{1}$$

Since $x \leq_T \Phi_i^x$ is Σ_3^0, $\varphi(x)$ is Π_4^0. Furthermore, $\varphi(f)$ if and only if $deg(f) = \mathbf{d}$. So, say $g \in \mathbf{d}$ is 4-generic, then there is $\sigma \subset p$ such that $\sigma \Vdash \varphi(x)$. But then for all $f \supset \sigma$ $\varphi(f)$ and we can choose such f to be Turing incomparable with g. Thus $\mathbf{d}$ is not treeable. □

Proposition 11. *For every computable ordinal* α*, there is a hyperarithmetic degree* $\mathbf{d}$ *with* $\mathbf{d} \geq \mathbf{0}^{(\alpha)}$ *that is not a degree of categoricity.*

Proof. Let $\psi(x)$ be given similar to $\varphi(x)$ as defined in Eq. (1) but replace x with $x^{(\alpha)}$. Let $\mathbf{c}$ be a hyperarithmetic $(\alpha+4)$-generic degree, then $\mathbf{c}^{(\alpha)}$ is not a degree of categoricity as it is not treeable by the same argument as in Lemma 10 with $\psi(x)$ in place of $\varphi(x)$. □

Note that by Eq. (1) in the proof of Lemma 10 every treeable degree is a countable Π_4^0 class. Tanaka [Tan72] showed that every countable Σ_{n+1}^0 class contains a Π_n^0 singleton. So, in particular, if $\mathbf{d}$ is treeable, then it contains a Π_4^0 singleton.

Proposition 12. *Every degree of categoricity contains a* Π_4^0 *singleton.*

2. Degrees of categoricity in $[\mathbf{0}', \mathbf{0}'']$

In this section we show that all degrees between $\mathbf{0}'$ and $\mathbf{0}''$ are degrees of categoricity. In [FKM10], Fokina, Kalimullin and Miller showed that every degree d.c.e. in and above $\mathbf{0}^{(n)}$ for some n is a degree of categoricity. They did this by noting that their proof that every d.c.e. degree is a degree of categoricity relativises to give that every degree d.c.e. in and above a given degree is a degree of categoricity *relative to the given degree*, and then using Marker extensions to get the desired result. Now that Csima and Ng have shown that every Δ_2^0 degree is a degree of categoricity in [CN22], we can verify that the methods of Fokina, Kalimullin and Miller also apply in this case, and obtain the desired result.

To begin, we give the definition of a categoricity spectrum relative to a degree.

Definition 4. Let τ be a computable vocabulary, let $\mathbf{c}$ be a Turing degree, let $\mathcal{A}$ be a $\mathbf{c}$-computable τ-structure, and let $(\mathcal{B}_e)_{e\in\omega}$ be an enumeration of all $\mathbf{c}$-computable τ-structures. The *categoricity spectrum of* $\mathcal{A}$ *relative to* $\mathbf{c}$ is the set

$$CatSpec_{\mathbf{c}}(\mathcal{A}) = \bigcap_{e\in\omega:\mathcal{B}_e\cong\mathcal{A}} \{deg(X) : (\exists f : \mathcal{A} \cong \mathcal{B}_e) X \geq_T f\}.$$

The proof given in Theorem 5.9 of [FKM10] shows that if $\mathbf{d} \geq_T \mathbf{0}^{(m)}$, and if there exists a $\mathbf{0}^{(m)}$-*acceptable* structure $\mathcal{M}$ with $CatSpec_{\mathbf{0}^{(m)}}(\mathcal{M})$ the upper cone above $\mathbf{d}$, then $\mathbf{d}$ is a strong degree of categoricity. A $\mathbf{0}^{(m)}$-*acceptable* structure is one with a $\mathbf{0}^{(m)}$-computable copy in which all predicates have an infinite computable subset. So it remains to argue that if $\mathbf{0}' < \mathbf{d} < \mathbf{0}''$, then there is a $\mathbf{0}^{(1)}$-*acceptable* structure $\mathcal{M}$ with $CatSpec_{\mathbf{0}^{(1)}}(\mathcal{M})$ the upper cone above $\mathbf{d}$.

We now consider the proof given in [CN22] that every Δ_2^0 degree is a strong degree of categoricity. It proceeds by considering a Δ_2^0 appoximation to a set $D \in \mathbf{d}$, an effective list of the possible computable structures, and builds the structures $\mathcal{A}$ and $\hat{\mathcal{A}}$ that witness $\mathbf{d}$ as a strong degree of categoricity in a stage-by-stage construction.

Though the proof as described in the paper uses infinitely many relation symbols, it is noted that this is just a convenience - the structure can be easily converted into a graph using standard codings. Running the construction relative to an oracle $\mathbf{0}'$, for a degree $\mathbf{d} \geq \mathbf{0}'$ that is Δ^0_2 in $\mathbf{0}'$ (i.e., $\mathbf{0}' \leq \mathbf{d} \leq \mathbf{0}''$), and adjoining the standard copy of a structure with strong degree of categoricity $\mathbf{0}'$ then yields a structure with strong degree of categoricity $\mathbf{d}$ relative to $\mathbf{0}'$. Moreover, when we transform the presentation into a graph, we easily obtain the infinite computable subset of the edge relation required to guarantee an acceptable copy of the structure - they occur as we mark off the independent modules where the different requirements will act. Combining this with Corollary 8 and Csima and Ng's result we obtain the following.

Theorem 13. *Every degree* $\mathbf{d}$ *with* $\mathbf{0}^{(\alpha)} \leq \mathbf{d} \leq \mathbf{0}^{(\alpha+1)}$ *for some computable ordinal* α *is the degree of categoricity of a computable structure.*

3. Not the degree of categoricity of a rigid structure

Corollary 8 shows that for $\alpha > 2$, every degree $\mathbf{d}$ between $\mathbf{0}^{(\alpha)}$ and $\mathbf{0}^{(\alpha+1)}$ is the degree of categoricity of a rigid structure. Bazhenov and Yamaleev showed that there is a 2-c.e. degree that is not the degree of categoricity of a rigid stucture [BY17]. In this section we prove the following theorem.

Theorem 14. *There is a degree* $\mathbf{d}$, $\mathbf{0}' < \mathbf{d} < \mathbf{0}''$, *that is not the degree of categoricity of a rigid structure.*

Theorem 14 shows a limitation to Turetsky's technique of coding paths through trees into the automorphisms of a structure: If one were to code treeable degrees below $\mathbf{0}''$, one would not be able to preserve the 1-1 correspondence between automorphisms and paths through the tree.

The main ingredient of the proof of Theorem 14 is the following lemma. Its proof combines Bazhenov and Yamaleev's construction with a true stage argument.

Lemma 15. *There exists a degree* $deg(D)$, $\mathbf{0}' < deg(D) < \mathbf{0}''$ *that satisfies the following: for all computable structures* $\mathcal{A}$ *and* $\mathcal{B}$ *and all* Δ^0_3 *isomorphisms* $g : \mathcal{A} \cong \mathcal{B}$, $deg(D) = deg(g)$ *implies that there is a computable structure* $\mathcal{N}$ *and an isomorphism* $f : \mathcal{A} \cong \mathcal{N}$ *with* $deg(f) \not\leq deg(D)$.

Notice that Lemma 15 already shows that there is a degree $\mathbf{d} \in (\mathbf{0}', \mathbf{0}'')$ that is not the strong degree of categoricity of any rigid structure.

Proof of Lemma 15. Let $g \leq \mathbf{0}''$, then g is limit computable in $\mathbf{0}'$. Assume that we have an approximation to $\emptyset'$ in terms of finite strings σ_i where

$$|\sigma_i| = \min\{\mu x[x \searrow \emptyset'_i], i\} \quad \text{and} \quad (\sigma_i(x)\downarrow \Rightarrow \sigma_i(x) = \emptyset'_i(x)).$$

Let the triple $(\mathcal{A}, \mathcal{B}, g)$ consist of computable graphs $\mathcal{A}$, $\mathcal{B}$ and a $\Delta^0_2(\emptyset')$ function g. Then we may approximate this triple via the triples $(\mathcal{A}[s], \mathcal{B}[s], g[s])$ where $\mathcal{A}[s]$ and $\mathcal{B}[s]$ are substructures of $\mathcal{A}$, respectively $\mathcal{B}$, with universe $\{0, \dots, s\}$, and, say that g is limit computable via $\Phi_e^{\emptyset'}$, then $g[s](x) = \Phi_e^{\sigma_s}(x, t)$ where $t = \max\{r < s : \Phi_{e,s}^{\sigma_s}(x, r)\downarrow\}$. Note that it is not necessarily the case that $\lim_{s\to\infty} g[s] = g$. However, if $i_0, \dots$ is a sequence of true stages in our approximation to $\emptyset'$, then $\lim_{s\to\infty} g[i_s] = g$. We will work with an enumeration of the approximations of all such triples $(\mathcal{A}_e, \mathcal{B}_e, g_e)_{e\in\omega}$. Our construction will build a set D, $\emptyset' \leq_T D \leq_T \emptyset''$ and ensure that if $g_e : \mathcal{A}_e \cong \mathcal{B}_e$ and $g_e \equiv_T D$, then there is a computable structure

$\mathcal{N}_e \cong \mathcal{A}_e$ and an isomorphism $f_e : \mathcal{N}_e \cong \mathcal{A}_e$ such that $f_e \not\leq_T D$. That is, for every e we aim to satisfy the following infinite set of requirements

$$S_{e,i}: \quad \left(g_e : \mathcal{A}_e \cong \mathcal{B}_e \wedge g_e \in \Delta^0_3 \wedge D = \Phi_e^{g_e} \wedge g_e = \Theta_e^D\right) \Rightarrow f_e \neq \Phi_i^D$$

and the global requirement

$$G : \emptyset' \leq_T D.$$

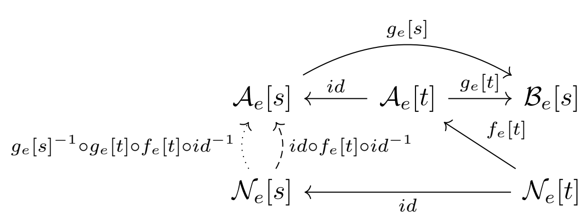
The requirement G is satisfied by building $D = \hat{D} \oplus \emptyset'$.

Given $\sigma, \tau \in 2^{<\omega}$ we say that τ is *σ-true*, if $\tau \preceq \sigma$. We call a stage t an s-true stage if the stage t approximation σ_t is σ_s-true. The key difference of our construction to Bazhenov and Yamaleev's construction is that at stage s we only consider the work of strategies acting at σ_s-true stages to build D.

For people who have seen that construction it might be easy to see that satisfying $\mathcal{S}_{e,i}$ using true stages gives a set that is $\Delta^0_2(\emptyset')$ and can not be the degree of categoricity of a rigid structure.

The difficulty arises from coding $\emptyset'$. The key argument for the coding is that if we take $D = \hat{D} \oplus \emptyset'$, then the computation Ψ_i^D might only change finitely many times at any x given that we preserve finite injury of $\hat{D}$ on the true path. This is because at some point in the approximation of $\emptyset'$ no bit lower than the use of $\Psi_i^D(x)$ will change anymore. Thus, if at that point we have ensured that on the true path $\Psi_i^D(x) \neq f_e$, then this will hold in the limit. However, with more complicated sets, such as $\emptyset''$ this is not true anymore.

Let $i_0, \dots$ be the sequence of true stages for $\emptyset'$. Together with $D = \lim_{j\to\omega} D[i_j]$ we will build $\mathcal{N}_e = \lim_{j\to\omega} \mathcal{N}_e[j]$. Say that a stage s in the construction is e-expansionary if $\mathcal{N}_e[s+1] \neq \mathcal{N}_e[s]$. An e-expansionary stage is always caused by some strategy that tries to satisfy $S_{e,i}$. Let s be an e-expansionary stage, then $\mathcal{N}_e[s]$ will have universe $\{0, \dots, s\}$ and will embed into $\mathcal{A}_e$. Say we are at some stage s, and $t < s$ is e-expansionary. The strategy acting for a requirement $S_{e,i}$ might force us to change the embedding $f_e[t]$ because $g_e[s] \neq g_e[t]$. There are two ways we can do this.



We can extend any partial embedding of $\mathcal{N}_e[s]$ into $\mathcal{A}_e[s]$ by recursively mapping elements not in the domain to the least element not in the range as $N_e[s] = A_e[s] = \{0, \dots, s\}$. We say that $f_e[s]$ is the *id-extension* of $f_e[t]$ if it is obtained by extending $id \circ f_e[t] \circ id^{-1}$ and $f_e[s]$ is a *g-extension* of $f_e[t]$ if it is obtained by extending $g_e[s]^{-1} \circ g_e[t] \circ f_e[t] \circ id^{-1}$. Note that we can take *g-extensions* because technically $g_e[t] : \mathcal{A}_e[t] \to \mathcal{B}_e[t]$ and $\mathcal{B}_e[t] \subseteq \mathcal{B}_e[s]$.

We use upper case greek letters to denote Turing operators and the corresponding lower case greek letters to denote their use. We abuse notation and do not write the oracle in the use function as the oracle will be implied from the context. As we are only interested in total functions we can make the following assumption on uses without loss of generality: If $\Phi^X(x)\downarrow$, then for all $y < x$, $\Phi^X(y)\downarrow$ and $\varphi(y) < \varphi(x)$.

During the construction we will use the following length agreement functions:

$$L(e)[s] = \max\Big\{x : \forall(y \leq x)\Big[\Phi_e^{g_e}(y)[s] = D[s](y) \wedge \forall(z < \varphi_e(y)[s])\left(\Theta_e^D(z)[s] = g_e(z)[s] \wedge (\{0, \dots, \varphi_e(y)[s]\} \subseteq \mathcal{A}_e[s])\right)\Big]\Big\}$$

$$l(e,i)[s] = \max\{x : \forall(y \leq x)(f_e(y)[s] = \Psi_i^D(y)[s])\}$$

The priority ordering on our requirements is given by $S_{e,i} < S_{e',i'}$ if and only if $\langle e,i\rangle < \langle e',i'\rangle$. If the above is true, we say that $S_{e,i}$ has higher priority than $S_{e',i'}$. Assume we are at stage s of the construction and let t be the last s-true stage. At the beginning of s we let $D[s] = D[t]$. A strategy $S_{e,i}$ might require attention because of the following reasons:

(1) Its witness $x_{e,i}$ is undefined.
(2) $x_{e,i} \notin D[s]$, $x_{e,i} < L(e)[s]$, and $f_e(y)$ is undefined for some $y < \varphi_e(x_{e,i})[s]$.
(3) $x_{e,i} \notin D[s]$, $x_{e,i} < L(e)[s]$, and $\varphi_e(x_{e,i})[s] < l(e,i)[s]$.
(4) $x_{e,i} \in D[s]$ and $x_{e,i} < L(e)[s]$.

Construction. At stage 0 define $D[0] = \emptyset$, $\mathcal{N}_e = \emptyset$ and $f_e = \emptyset$ for all $e \in \omega$. Say we are at stage $s > 0$ of the construction. Let $D[s] = D[t]$ and then enumerate $2x+1$ into $D[s]$ for all x such that $x \in \emptyset'_s$. Find the least requirement that requires attention and initialize all lower priority strategies. Then choose the following depending on what caused the attention.

(1) Define $x_{e,i}$ to be an even number larger than all numbers used in the construction so far.
(2) Let s^- be the last e-expansionary stage. Define $\mathcal{N}_e[s]$ and $f_e[s]$ as the *id*-extension of $f_e[s^-]$.
(3) Enumerate $x_{e,i}$ into $D[s]$.
(4) Extract $x_{e,i}$ from $D[s]$. Let s^- be the last e-expansionary stage. Define $\mathcal{N}_e[s]$ and $f_e[s]$ as the g extension of $f_e[s^-]$.

Verification. We first verify the construction for a single requirement $S_{e,i}$ in isolation. Say $S_{e,i}$ acts for the first time at stage s_0 and picks $x_{e,i}$. Notice that as $S_{e,i}$ will never be initialized it won't ever act again because of reason (1). Now, say that $s_1 > s_0$ is a true stage such that $\emptyset'_{s_0} \restriction x_{e,i}/2 = \emptyset' \restriction x_{e,i}/2$. By our isolation assumption the bits below $x_{e,i}$ in our approximation to D won't change anymore after s_1.

Say $S_{e,i}$ needs attention because of reason (2) at true stage $s_2 > s_1$. We do an *id*-extension of $f_e[s^-]$. Such an extension can always be done because $f_e : \mathcal{N}_e \to \mathcal{A}_e[s^-]$ and $\mathcal{A}_e[s^-] \subseteq \mathcal{A}_e[s_2]$. If $S_{e,i}$ never needs attention again we have satisfied $S_{e,i}$. Now, say that s_3 is a true stage such that $S_{e,i}$ receives attention. Then this can not be because of reason (2) because if $x_{e,i} < L(e)[s_3]$, then $\Theta_e^D(z)[s_3] = \Theta_e^D(z)[s_2] = g_e[s]$ for all $z < \varphi_e(x_{e,i})[s]$ and so $\varphi_e(x_{e,i})[s_2] = \varphi_e(x_{e,i})[s_3]$ and we have already extended f_e at s_2. Thus $S_{e,i}$ must act because of reason (3) and so $x_{e,i}$ is enumerated into D at stage s_3. Notice that, since s_3 is a true stage, for every $t \geq s_3$, s_3 is t-true. Let s_4 be the next true stage such that $S_{e,i}$ receives attention. Notice that this must be because of reason (4) as $x_{e,i} \in D[s_4]$. We extract $x_{e,i}$ from D and do a g-extension hoping to diagonalize against Ψ_i^D computing f_e. It might be the case that there is a stage t (not necessarily a true stage), $s_3 < t < s_4$ such that $t = s_4^-$, $g_e[t] \subseteq g_e[s_4]$ and the g-extension of $f_e[t]$ is an *id*-extension. Say $S_{e,j}$ does the g-extension at t, then $S_{e,j}$ is not of higher priority than $S_{e,i}$ as otherwise $S_{e,i}$ would have been reset. But then $L(e)[t] > x_{e,j} > x_{e,i}$ and thus $S_{e,j} = S_{e,i}$. So, after doing the g-extension at s_4 we have that

$$f_e[s_4] \supseteq f_e[t] \neq f_e[s_3] = \Psi_i^D[s_3] \subseteq \Psi_i^D[s_4].$$

So we have successfully diagonalized and given that D does not change below $x_{e,i}$ anymore $\Psi_i^D \neq f_e$.

We have that $x_{e,i} > L(e)[s_4 + 1]$. This allows g_e to change and achieve that $L(e)[s_5] > x_{e,i}$ at some stage s_5. The risk is that $\varphi_e(x_{e,i})[s_5] < \varphi_e(x_{e,i})[s_4]$ and that thus we might want to take action because of (3) again. Notice that this would imply that g_e changed below $\varphi_e(x_{e,i})[s_4]$. But at s_4 we had that

$$\forall (z < \varphi_e(x_{e,i})[s_4])(\Theta_e^D(z)[s_4] = g_e(z)[s_4]).$$

So this is particularly true for all $z < \varphi_e(x_{e,i})[s_5]$. By our assumption on use functions φ that $x < y$ implies $\varphi(x) < \varphi(y)$ we have that

$$\theta_e(\varphi_e(x_{e,i})[s_5]) < \theta_e(\varphi_e(x_{e,i})[s_4]) \leq x_{e,i}$$

and as $D[s_4] \restriction x_{e,i} = D[s_5] \restriction x_{e,i}$,

$$\forall (z < \varphi_e(x_{e,i})[s_5])(\Theta_e^D(z)[s_4] = g_e(z)[s_5] = g_e(z)[s_4])$$

contradicting that g_e changed below $\varphi_e(x_{e,i})$. Notice that a true stage t is s-true for all $s > t$. Thus, in isolation, $S_{e,i}$ will never require attention after stage s_4.

We now drop the isolation assumption. Notice first that if $t > s_4$ and $f_e[t] \restriction \varphi_e(x_{e,i})[s_4] \neq f_e[s_4] \restriction \varphi_e(x_{e,i})[s_4]$, then by the argument in the above paragraph this can only have happened because of a g-extension of a higher priority argument which would cause $S_{e,i}$ to initialize. Let us show that no requirement is initialized infinitely often. Recall that in isolation every requirement requires attention at most 4 times on the true path. Using the fact that every true stage t is s-true for all $t > s$ we can see by induction that for every requirement there is a stage after which it is not initialized again. By the same argument, there is a stage t such that if $S_{e,i}$ requires attention at $s > t$, then it is the highest priority requirement needing attention and thus will receive attention at s.

Notice, that if the lefthand side of the implication in $S_{e,i}$ is true, then $\mathcal{N}_e$ is necessarily infinite and isomorphic to $\mathcal{A}_e$ as for every i, $S_{e,i}$ receives attention because of condition (2) at least once. Furthermore, no Ψ_i^D can compute f_e, by our above argument. So, every requirement $S_{e,i}$ is satisfied. The requirement G is trivially satisfied in the limit. □

Lemma 16. *A degree* $\mathbf{d}$ *is a degree of categoricity of a rigid structure if and only if it is the strong degree of categoricity of a rigid structure.*

Proof. The direction from right to left is trivial. For the other direction recall that the spectral dimension of a rigid structure is finite [BKY16, Theorem 3.1]. So say $\mathcal{A}$ is a rigid structure with spectral dimension α, $1 < \alpha < \omega$ and consider the cardinal sum $\bigotimes \mathcal{A}$, i.e., if τ is the vocabulary of $\mathcal{A}$, then $\tau^+ = \tau \cup \{R_i/1 :\in \omega\}$ is the vocabulary of $\bigotimes \mathcal{A}$, and every column $R_i^{\bigotimes \mathcal{A}}$ is isomorphic to a copy of $\mathcal{A}$. $\bigotimes \mathcal{A}$ is clearly computably presentable and rigid. Furthermore, if $\mathcal{A}$ has degree of categoricity $\mathbf{d}$, then there are computable copies $\mathcal{A}_{n,1}$, $\mathcal{A}_{n,2}$, $n < \alpha$ and unique isomorphisms $f_n : \mathcal{A}_{n,1} \to \mathcal{A}_{n,2}$ such that $\bigoplus deg(f_n) = \mathbf{d}$. Consider the structures

$(\bigotimes \mathcal{A})_1 \cong (\bigotimes \mathcal{A})_2 \cong \bigotimes \mathcal{A}$ such that $n \mapsto \langle i, n \rangle$ are uniformly computable isomorphisms between $\mathcal{A}_{(i \mod \alpha),j}$ and the ith column of $(\bigotimes \mathcal{A})_j$. Then $(\bigotimes \mathcal{A})_1, (\bigotimes \mathcal{A})_2$ witness that **d** is the strong degree of categoricity of a rigid structure. □

Since Lemma 16 shows that every degree of categoricity of a rigid structure is the strong degree of categoricity of a rigid structure and Lemma 15 shows that there is a degree that is not the strong degree of a rigid structure we immediately get Theorem 14.

References

[AC12] Bernard A. Anderson and Barbara F. Csima. "Degrees That Are Not Degrees of Categoricity". 2012. arXiv: `1210.4220`.

[BKY16] Nikolay Alekseevich Bazhenov, I. Sh Kalimullin, and Mars Mansurovich Yamaleev. "Degrees of Categoricity vs. Strong Degrees of Categoricity". In: *Algebra and Logic* 55.2 (2016), pp. 173–177.

[BKY18] Nikolay A. Bazhenov, Iskander Sh Kalimullin, and Mars M. Yamaleev. "Degrees of Categoricity and Spectral Dimension". In: *The Journal of Symbolic Logic* 83.1 (2018), pp. 103–116.

[BY17] Nikolay A. Bazhenov and Mars M. Yamaleev. "Degrees of Categoricity of Rigid Structures". In: *Conference on Computability in Europe.* Springer, 2017, pp. 152–161.

[CFS+13] Barbara F. Csima, Johanna NY Franklin, Richard A. Shore, et al. "Degrees of Categoricity and the Hyperarithmetic Hierarchy". In: *Notre Dame Journal of Formal Logic* 54.2 (2013), pp. 215–231.

[CN22] Barbara F. Csima and Keng Meng Ng. "Every Δ^0_2 Degree Is a Strong Degree of Categoricity". In: *Journal of Mathematical Logic* (2022).

[Csi+18] Barbara F. Csima et al. "Degrees of Categoricity above Limit Ordinals". In: *Computability* (Preprint 2018), pp. 1–11.

[FKM10] Ekaterina Fokina, Iskander Kalimullin, and Russell Miller. "Degrees of Categoricity of Computable Structures". In: *Archive for Mathematical Logic* 49.1 (2010), pp. 51–67.

[Fra17] Johanna NY Franklin. "Strength and Weakness in Computable Structure Theory". In: *Computability and Complexity.* Springer, 2017, pp. 302–323.

[FS14] Johanna NY Franklin and Reed Solomon. "Degrees That Are Low for Isomorphism". In: *Computability* 3.2 (2014), pp. 73–89.

[FS56] Albrecht Fröhlich and John C. Shepherdson. "Effective Procedures in Field Theory". In: *Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences* 248.950 (1956), pp. 407–432.

[Gon80] S. S. Goncharov. "Problem of the Number of Non-Self-Equivalent Constructivizations". In: *Algebra and Logic* 19.6 (1980), pp. 401–414.

[Har76] Leo Harrington. *Arithmetically Incomparable Arithmetic Singletons.* 1976.

[JM69] Carl Jockusch and Thomas McLaughlin. "Countable Retracing Functions and Π^0_2 Predicates". In: *Pacific Journal of Mathematics* 30.1 (1969), pp. 67–93.

[Mal62] A. I. Maltsev. "On Recursive Abelian Groups". In: *Soviet Mathematics. Doklady.* Vol. 3. 1962.

[Odi99] P Odifreddi. *Classical Recursion Theory, Volume II*. Vol. 143. Studies in Logic and the Foundations of Mathematics. 1999.

[Sac90] Gerald E. Sacks. *Higher Recursion Theory*. Vol. 2. Cambridge University Press, 1990.

[Tan72] Hisao Tanaka. "A Property of Arithmetic Sets". In: *Proceedings of the American Mathematical Society* 31.2 (1972), pp. 521–524.

[Tur20] Dan Turetsky. "Coding in the Automorphism Group of a Computably Categorical Structure". In: *Journal of Mathematical Logic* 20.03 (2020). ISSN: 0219-0613. DOI: 10.1142/S0219061320500166. arXiv: 1905.07850.

Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario
Email address: csima@uwaterloo.ca

Department of Mathematics, University of California, Berkeley and Institute of Discrete Mathematics and Geometry, Technische Universität Wien
Email address: dino@math.berkeley.edu